Statistical Methods 2004, 6(1), 82-89

Estimation of Location and Scale Parameters in a Two -Parameter Exponential Distribution from a Censored Sample

R. P. Suresh
Indian Institute Of Management
Kozhikode, Kunnamangalam P.O.
Calicut 673571India.

Abstract: In this paper, we consider estimating the location and scale parameters in the two-parameter Exponential Distribution using a Type II censored sample. We derive the Modified Maximum Likelihood Estimators using the approach of Tiku and Suresh (1992) as the likelihood equations are intractable. We compare these estimators with other existing estimators, and also study their properties. We derive a test for testing the equality of the scale parameter.

Key Words and Phrases: Modified Maximum Likelihood Estimators, Type II censored sample

1. Introduction

Censoring is quite common in the study of Reliability and life testing. Type II censored sample occurs when a certain number (or a proportion) of observations are censored on the left or right or both. For example, in early childhood learning centres, interest often focuses upon testing children to determine when a child learns to accomplish certain specified tasks. The age at which a child learns the task would be considered the time-to-event. Often, some children can already perform the task when they start their study. Such event times are considered left censored. Some children undergoing testing, may not learn the task during the entire study period, in which case such event times would be right censored. Thus, the sample would also be doubly censored.

Consider a doubly censored sample $Y_{r+1} \le Y_{r+2} \le \ldots \le Y_{n-s}$ (with r observations censored on the left and s observations censored on the right, where $r = [nq_1] + 1$, and $s = [nq_2] + 1$) from a two parameter Exponential distribution with density given by

$$f(x \mid \theta, \sigma) = (1/\sigma)\exp[-(x-\theta)/\sigma],\ x>\theta,\ \sigma>0 \tag{1.1}$$

The likelihood of the sample is given by

$$L = [F(Y_{r+1})]^r [1-F(Y_{n-s})]^s \prod_{i=r+1}^{n-s} f(Y_i)$$

$$i.e.,\ L = [1-\exp(-Z_{r+1})]^r \exp(-s.\,Z_{n-s})\,(1/\sigma)^{n-s-r}\exp(-\sum_{r+1}^{n-s} Z_i),$$

where

$$Z_i = (Y_i - \theta)/\sigma,\ i = r+1, \ldots n-s.$$

The log likelihood of the sample is given by

$$\text{Log } L = r\ln[1-\exp(-Z_{r+1})] - s\,Z_{n-s} + (n-s-r)\ln(\sigma) - \sum_{r+1}^{n-s} Z_i$$

The likelihood equations are given by

$$\frac{\partial \ln L}{\partial \theta} = \frac{r}{\sigma}\{1 - \frac{1}{1-\exp(-Z_{r+1})}\} + \frac{n-r}{\sigma} = 0 \tag{1.2}$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{n-r-s}{\sigma} + r\frac{Z_{r+1}}{\sigma}\{1-\frac{1}{1-\exp(-Z_{r+1})}\} + s\frac{Z_{n-s}}{\sigma} + \frac{\sum Z_i}{\sigma} = 0 \tag{1.3}$$

The ML equations (1.2) & (1.3) do not have explicit solution for θ and σ. This is due to the fact that the term $g(z) = 1/(1-e^{-z})$ is intractable. In this paper, we use the Modified Maximum Likelihood approach to derive approximate MLE's for θ

and σ by linearizing the term $g\ (Z_{r+1}) = \dfrac{1}{1-\exp(-Z_{r+1})}$ using Taylor series expansion around the quantile point of F. Similar procedures were earlier used by different researchers for estimating the parameters for Normal distribution, Logistic distribution and Extreme value distribution (See Balakrishnan and Cohen, 1991).

In Section 2 of the paper, we derive new estimators and compare these estimators with existing estimators for small samples. In Section 3 of the paper, we derive a test statistic for testing the hypothesis about the scale parameter.

2. MML Estimation for θ and σ

Modified Maximum Likelihood (MML) estimation is based on linearizing the intractable terms in Likelihood equations after expressing these terms in terms of order statistics. The linearization is done in such a way that the derived MML estimators retain all the desirable asymptotic properties of the ML estimators but have the additional advantage that they have explicit expressions for small samples. Tiku and Suresh (1992) used the Taylor series expansion of the intractable terms in estimating the location and scale parameters in a symmetric family of distributions, which includes a number of well-known distributions such as Normal, Student's t etc. They also showed that the MML estimators, thus derived, are asymptotically fully efficient, and almost fully efficient for small samples (see also Bhattacharya (1985), Tiku, Tan and Balakrishnan (1986), Vaughan (1992) and Suresh (1997)).

In the following, we derive the MML estimators for θ and σ. First, we linearize the term $g\ (Z_{r+1})$ using Taylor Series expansion around λ_{q_1}, the quantile of F at q_1, and truncating the series at the linear term, where F is the cumulative distribution function corresponding to the density in (1.1) with $\theta = 0$ and $\sigma = 1$. The approximation is given by

$$g(Z_{r+1}) \cong g(\lambda_{q_1}) + (Z_{r+1} - \lambda_{q_1})\, g'(\lambda_{q_1})$$

where $g'(\lambda_{q_1})$ is the derivative of g at λ_{q_1}. After simplifying we get,

$$g(Z_{r+1}) \cong a - bZ_{r+1} \tag{2.1}$$

where $a = 1/q_1 + \lambda_{q_1}(1 - q_1)/q_1^2$, and $b = (1 - q_1)/q_1^2$.

Substituting the above approximation for $g(Z_{r+1})$ in the Likelihood equations (1.2) and (1.3), we get the modified equations given by

$$\frac{\partial \ln L}{\partial \theta} \cong \frac{\partial \ln \tilde{L}}{\partial \theta} = \frac{r}{\sigma}\{1-(a - bZ_{r+1})\} + \frac{n-r}{\sigma} = 0 \tag{2.2}$$

$$\frac{\partial \ln L}{\partial \sigma} \cong \frac{\partial \ln \tilde{L}}{\partial \sigma} = -\frac{n-r-s}{\sigma} + r\frac{Z_{r+1}}{\sigma}\{a - bZ_{r+1}\} + s\frac{Z_{n-s}}{\sigma} + \frac{\sum Z_i}{\sigma} = 0 \tag{2.3}$$

Solving these equations, we get the Modified Maximum Likelihood estimators and are given by

$$\hat{\theta} = Y_{r+1} + [(n - ar)/br]\hat{\sigma} \tag{2.4}$$

and

$$\hat{\sigma} = [\sum_{r+1}^{n-s} Y_i + sY_{n-s} - (n-r)Y_{r+1}]/(n-r-s) \tag{2.5}$$

It may be noted that the approximation in (2.1) will be a strict equality when n is large (see Tiku and Suresh(1992) and Bhattacharya(1985)). In view of this, the modified Likelihood equations $\frac{\partial \ln \tilde{L}}{\partial \theta}$ and $\frac{\partial \ln \tilde{L}}{\partial \sigma}$ coincide with $\frac{\partial \ln L}{\partial \theta}$ and $\frac{\partial \ln L}{\partial \sigma}$ asymptotically. Hence, the estimators $\hat{\theta}$ and $\hat{\sigma}$ derived here are asymptotically equivalent to MLEs of θ and σ, respectively, and thus, are asymptotically fully efficient. Note that both $\hat{\theta}$ and $\hat{\sigma}$ are linear functions of order statistics. It may be noted that both the estimators $\hat{\theta}$ and $\hat{\sigma}$ are biased estimators with expectations given by

$$\mathrm{E}\,(\hat{\theta}) = \theta + [(n-ar)/br].(\sigma/(n-r-s)) \text{ and } \mathrm{E}\,(\hat{\sigma}) = \sigma(1-1/(n-r-s)).$$

Remark 2.1: Tiku (1967) derived estimators of location and scale parameters by approximating g(z) by $\alpha+\beta z$, where α and β are derived such that the expectation of the approximate Likelihood equations are zeros, viz., $\mathrm{E}(\frac{\partial \ln \tilde{L}}{\partial \theta})=0$ and $\mathrm{E}(\frac{\partial \ln \tilde{L}}{\partial \sigma})=0$. However, there are no explicit expressions for the coefficients α and β, thus derived. It may be noted that the expression for $\hat{\sigma}$ derived here coincides with that of Tiku's estimator while $\hat{\theta}$ is different from their estimator. Tiku(1967) compared the estimator $\hat{\sigma}$ with the Best Linear Unbiased Estimator (BLUE) for σ for small samples, and proved that $\hat{\sigma}$ has smaller variance as compared to BLUE.

Here, we compare the estimators of the location parameters $\hat{\theta}_{proposed}$, $\hat{\theta}_{Tiku}$ and $\hat{\theta}_{BLUE}$ in terms of their variances. Tables 2.1-2.4 give the variances of the estimators for different values of n.

From the tables 2.1-2.4, it is clear that the proposed estimator is highly efficient as compared to Tiku's estimator and is remarkably efficient as compared to BLUE estimator, for small samples. Since the bias in the proposed estimator and Tiku's estimator are very small for large samples and moderate censoring, the estimators $\hat{\theta}_{proposed}$ and $\hat{\theta}_{Tiku}$ have smaller variances than the BLUE.

Table 2.1: Comparison of estimators of θ for n = 10

q_1	q_2	MSE ($\hat{\theta}_{proposed}$)	MSE ($\hat{\theta}_{Tiku}$)	Var ($\hat{\theta}_{BLUE}$)
.1	0	.007729	.014967	.0223457
.1	.2	.007436	.017641	.0223457
.1	.4	.0070288	.024347	.0223457
.2	0	.0097925	.03056707	.03797068
.2	.2	.00942314	.04351468	.03797068
.2	.4	.00957782	.08671071	.0379068
.3	0	.01307135	.06827547	.05837884
.3	.2	.01370617	.14550824	.05837884
.3	.4	.0197915	.11873761	.05837884

Table 2.2: Comparison of estimators of θ for n = 20

q_1	q_2	MSE($\hat{\theta}_{proposed}$)	MSE ($\hat{\theta}_{Tiku}$)	Var ($\hat{\theta}_{BLUE}$)
.1	0	.0020432	.00350916	.0083565
.1	.2	.00196187	.00395088	.0083565
.1	.4	.00184724	.00494265	.0083565
.2	0	.00252218	.00690868	.01572296
.2	.2	.0024163	.00918684	.01572296
.2	.4	.0024374	.01672755	.01572296
.3	0	.0032747	.01474887	.02526945
.3	.2	.00342175	.02797558	.02526945
.3	.4	.00494217	.04393423	.02526945

Table 2.3: Comparison of estimators of θ for n = 30

q_1	q_2	MSE ($\hat{\theta}_{proposed}$)	MSE ($\hat{\theta}_{Tiku}$)	Var ($\hat{\theta}_{BLUE}$)
.1	0	.00094878	.00156457	.00494742
.1	.2	.0009102	.0017348	.00494742
.1	.4	.0008539	.00210819	.00494742
.2	0	.00114763	.00300168	.00976283
.2	.2	.00109704	.00389865	.00976283
.2	.4	.00110123	.00678895	.00976283
.3	0	.00146704	.00625759	.01598687
.3	.2	.00153028	.0113536	.01598687
.3	.4	.00220266	.02178625	.01598687

Table 2.4: Comparison of estimators of θ for n = 50

q_1	q_2	MSE ($\hat{\theta}_{proposed}$)	MSE ($\hat{\theta}_{Tiku}$)	Var ($\hat{\theta}_{BLUE}$)
.1	0	.00035994	.0005745	.00266963
.1	.2	.00034502	.00062934	.00266963
.1	.4	.00032362	.00074724	.00266963
.2	0	.00042414	.00106825	.00551377
.2	.2	.00040458	.00136137	.00551377
.2	.4	.00040404	.00228127	.00551377
.3	0	.00053393	.00217547	.00918215
.3	.2	.00055549	.00380646	.00918215
.3	.4	.000796	.00833297	.00918215

3. Test statistic for the scale parameter

Note that $(n - r - s)\hat{\sigma}$ can be expressed as

$$(n-r-s)\hat{\sigma} = (\sum_{r+1}^{n-s} Y_i + sY_{n-s} - (n-r)Y_{r+1}) = \sum_{r+2}^{n-s} (n-j+1)(Y_j - Y_{j-1})$$

It may be noted that $(n - j + 1)(Y_j - Y_{j-1})$, $j = r + 2, \ldots, n - s$ form the normalized spacing from Exponential distribution with scale parameter σ, and hence, are independent and identically distributed as Exponential distribution with scale parameter σ. Hence, it follows that the estimator $(n - r - s)\hat{\sigma}$ derived here can be written as sum of $n - r - s - 1$ i.i.d. random variables, each with Exponential Distribution, and hence, $(n - r - s)\hat{\sigma}$ has a Gamma distribution with parameter $n - r - s - 1$ and σ. In view of the above, it follows from Central Limit Theorem for i.i.d. random variables that asymptotically, $\dfrac{(n-r-s)\hat{\sigma} - (n-r-s-1)\sigma}{\sqrt{n-r-s-1}\sigma}$ $\approx N(0,1)$.

Hence, we can derive asymptotic test for testing H_0: $\sigma = \sigma_0$ as follows:

Reject H_0 for large values of the statistic $Z=\frac{(n-r-s)\hat{\sigma}-(n-r-s-1)\sigma_0}{\sqrt{n-r-s-1}\sigma_0}$, the

critical values can be obtained using the standard normal tables.

It can be shown that the asymptotic power of the test is unity.

Acknowledgement: The author is thankful to the referee for the useful comments, which has resulted in an improved version of the paper. The work is supported by a Small Grant Research Project from Indian Institute of Management Kozhikode.

References

1. **Balakrishnan, N. and Cohen, A.C. (1991).** *Order Statistics and Inference,* New York: Academic Press,.

2. **Bhattacharya, G.K. (1985).** The asymptotics of maximum likelihood and related estimators based on Type II censored data, *J. Amer. Statist. Assoc.,* 80, 398-404.

3. **Suresh, R.P. (1997).** On approximate likelihood estimators in censored normal samples, *Gujarat Statistical Review*, 24, 21-28.

4. **Tan, W.Y. (1985).** On Tiku's robust procedure – a Bayesian insight, *J.Statist. Plann. and Inf.*, 11, 329-340.

5. **Tiku, M.L. (1967).** Estimating the mean and standard deviation from a censored sample, *Biometrika,* 54, 155-165.

6. **Tikku, M.L. (1968).** Estimating the parameters of normal and logistic distribution from censored samples, *Austral. J. Statistic*. 10, 64-74.

7. **Tiku, M.L; Tan, W.Y. and Balakrishnan, N. (1986).** *Robust Inference*, Marvel Delker, Inc., New York.

8. **Tiku, M.L and Suresh, R.P. (1992).** A new method of estimation for location and scale parameters, *J. Statist. Plann. and Inf.*, 30, 281-292.

9. **Vaughan, V.C. (1992).** On Tiku-Suresh Method of Estimation, *Commun. Stat. Theory Meth.*, 21, 451-469.