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Miscellaneous Notes

ON THE INTER-RELATIONSHIPS BETWEEN SOME CLASSES OF LIFE DISTRIBUTIONS

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ABSTRACT: It is well known that if F is IFR, then it is also $DMRL$; however, the converse is not true, in general. Kupka and Loo (1989) proved that a distribution with convex decreasing mean residual life (CDMRL) function has an Increasing Failure Rate (IFR) distribution. In this paper, we show that this condition (viz., convexity of DMRL) is not necessary for F to be IFR. It is observed that contrary to the monotonic classes, in the non-monotonic classes, F is BFR does not imply that F is $IDMRL$. In this paper, we derive a result that gives an additional condition required for a BFR distribution to be an $IDMRL$ distribution.

Key words and phrases : Reliability theory, Failure rate, Mean residual life, IFR, DMRL, BFR, IDMRL

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1. Introduction

In reliability and life testing analysis, a number of non-parametric classes of life distributions are considered to model the life times of individuals as well as mechanical systems or components. Most of these classes characterize the ageing (positive, negative or no ageing) properties of the underlying phenomenon. Some of the most commonly used classes are the ones defined in terms of Failure Rate and Mean Residual Life functions. The increasing Failure Rate and Decreasing Mean Residual Life classes of distributions have been found quite useful in several studies in Reliability and life testing (see, e.g., Barlow and Proschan (1975)). For definitions and details about the classes of life distributions Increasing Failure Rate (IFR), Decreasing Mean Residual Life (DMRL), Bathtub shaped Failure Rate (BFR) and Increasing initially and then decreasing Mean Residual Life (IDMRL) we refer to Deshpande and Suresh (1990), Aarset (1985, 1987) and Guess, Hollander and Proschan (1986)).

It is well known that if a distribution is IFR, it will also be DMRL, but the converse is not true, in general (see Rolski (1975)). Kupka and Loo (1989) proved that a distribution with convex decreasing mean residual life (CDMRL) function has an Increasing Failure Rate (IFR) distribution. This raises the following question :

Whether the convexity of the MRL function is necessary and sufficient for distributions with DMRL function to have an IFR function?

In Section 2 of this paper, we show that this condition is not necessary.

Next, we consider the non-monotonic classes of life distributions viz., BFR and IDMRL distributions. Several authors have studied these classes of life distribtions (see, e.g., Aarset (1985, 1987) and Guess, Hollander and Proschan (1986), Rajarshi and Rajarshi (1988), Deshpande and Suresh (1990)). In Section 2 of this paper, we show that, contrary to the monotonic classes, in the non-monotonic classes, F is BFR does not imply that F is IDMRL. We also derive a result that gives an additional condition required for a BFR distribution to be an IDMRL distribution.

2. Main Results

In Subsection 2.1, we show that convexity of the DMRL function is not necessary for a distribution to be IFR, and in Subsection 2.2, we provide an additional condition for a BFR distribution to be IDMRL.

2.1 Convexity of DMRL is not necessary to be IFR Kupka and Loo (1989) proved that convexity of DMRL function of a distribution imply that the distribution is having IFR function. However, it is well known that a distribution with DMRL function need not be IFR. The above result, therefore, raises a question as to whether convexity of the DMRL function is necessary for a DMRL distribution to be an IFR distribution also? In the following, we provide an answer for this question.

Consider the function

$$m(t) = 13/16 - (3/4)t - (1/16)t^2, 0 \leq t \leq 1.$$
$$m(t) = 0 \quad \text{otherwise.} \tag{2.1}$$

Gupta and Kirmani (1998) provide necessary and sufficient conditions for a function $m(t)$ to be Mean Residual Life function of a non-negative random variable. It can be easily seen that $m(t)$ given in (2.1) satisfies all those conditions, and hence $m(t)$ is a Mean Residual Life function of a non-negative random variable. It may be noted that $m(t)$ is a concave decreasing function.

The failure rate function corresponding to the above MRL function is given by

$$\begin{aligned} r(t) &= (1 + m'(t))/m(t) \\ &= (1/4 - t/8)/(13/16 - (3/4)t - t^2/16), 0 < t < 1. \end{aligned}$$

It can be easily seen that $r(t)$ is increasing in (0, 1), and hence F is IFR. Thus, convexity of a decreasing mean residual life function is not necessary for DMRL distribution to be IFR.

2.2 Characterization in the class of IDMRL distribution

In the study of monotone classes of life distributions, it is well known that F is IFR implies that F is DMRL, implying essentially that whenever a random phenomenon exhibits positive ageing behaviour in terms of the failure rate function, it will exhibit positive ageing property in terms of Mean Residual Life function also. However, in the non-monotone classes of life distributions, such an implication does not hold good. For example, consider the following MRL function (due to Muth (1977))

$$m(t) = 1/(1 + 2.3t^2), t \geq 0. \tag{2.2}$$

It is easy to show that the failure rate function corresponding to the above MRLF is bathtub shaped, and hence F is BFR. Clearly F is DMRL and not IDMRL. The result that the non-monotonic ageing property exhibited in terms of the Failure rate need not be not carried through to the MRL function is not counter-intuitive, as the failure rate is an instantaneous property while MRL function takes into account the entire residual life.

It is of interest to study the nature of the MRL function when F is BFR. In this section, we provide an additional condition required for a BFR distribution to be IDMRL.

It is easily observed that when F is BFR, the mean residual life will decrease eventually, indicating that there will be positive effect of ageing, after certain age. This result is stated in the following Lemma.

Lemma 2.1 *Let F be BFR with change-point t_0. Then the mean residual life function $m(t)$ is decreasing for $t \geq t_0$.*

The following result follows easily from the above Lemma.

Corollary 2.1 *If F is BFR with change-point τ and F is IDMRL with change-point ε, then $\varepsilon \leq \tau$.*

Theorem 2.1 *Let F be continuous on $[0,\infty)$ and twice differentiable on $(0,\infty)$. Let F be BFR. Let $m(t)$ be the mean residual life function of F.*

(i) *If there exists a y_0, such that $m'(y_0) = 0$, then F is IDMRL*

(ii) *Otherwise, namely, if there does not exist a y_0 such that $m'(y_0) = 0$, then F is DMRL.*

Proof. The failure rate function corresponding to F is given by $r(t) = \frac{f(t)}{\bar{F}(t)} > 0$. The MRL function is given by

$$m(t) = \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(u)du, t \geq 0.$$

It follows that $m(t)$ is continuous, twice differentiable on $(0,\infty)$.
We have $m'(t) = \frac{1}{\bar{F}(t)}(-\bar{F}(t)) + \int_t^\infty \bar{F}(x)dx(-1/(\bar{F}(t))^2(-f(t))$
i.e., $m'(t) = m(t).r(t) - 1$. (2.3)

$$\begin{aligned}\text{Also, } m'(t) &= \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} r(t)dx - 1 \\ &= \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)}[r(t) - r(x)]dx + \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} r(x)dx - 1.\end{aligned}$$

But $\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} r(x)dx = \int_t^\infty \frac{f(x)}{\bar{F}(t)}dx = \frac{\bar{F}(t)}{\bar{F}(t)} = 1$.
Hence $m'(t) = \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)}[r(t) - r(x)]dx$. (2.4)
Notice that since F is differentiable twice, $r(t) = \frac{f(t)}{\bar{F}(t)}$ is differentiable once on $(0,\infty)$.
Hence F is BFR $\Leftrightarrow$ there exists a $t_0 > 0$ such that

$$\left.\begin{array}{c} r'(t) < 0 \text{ for } t < t_0 \\ r'(t_0) = 0 \\ r'(t) > 0 \text{ for } t > t_0 \end{array}\right\} \tag{2.5}$$

Case (i) : There exists a $y_0 > 0$ such that $m'(y_0) = 0$.

We claim here that $m''(y_0) < 0$. Note that since $m'(y_0) = 0$, by differentiating (2.4) at y_0, we get

$$\begin{aligned} m''(y_0) &= m'(y_0)r(y_0) + r'(y_0)m(y_0) \\ &= r'(y_0)m(y_0). \end{aligned}$$

Thus, $m''(y_0) < 0 \Leftrightarrow r'(y_0) < 0$ i.e., $m''(y_0) < 0$ if and only if $y_0 < t_0$
Now, let us show that $y_0 < t_0$. Assume, on the contrary, that $y_0 \geq t_0$.
From (2.5), it follows that

$r(t_1) < r(t_2), t_0 \leq t_1 < t_2$.

Hence, $m'(t) = \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)}[r(t) - r(x)]dx < 0$ for $t \geq t_0$ i.e., $m'(y_0) < 0$ which is a contradiction.
Hence $y_0 < t_0$ and $m''(y_0) < 0$, hence the claim is proved.
Now, we prove that there exists a unique y such that $m'(y) = 0$.
Suppose there exists $y_1 \neq y_0$ such that $m'(y_1) = 0$. It follows from above that $y_1 < t_0$.
Let $y_1 < y_0 < t_0$. Then, we have.

$$\begin{aligned} m'(y_1) &= \int_{y_1}^{\infty} \frac{\bar{F}(y)}{\bar{F}(y_1)}[r(y_1) - r(y)]dy \\ &= \int_{y_1}^{y_0} \frac{\bar{F}(y)}{\bar{F}(y_1)}[r(y_1) - r(y)]dy + \int_{y_0}^{\infty} \frac{\bar{F}(y)}{\bar{F}(y_1)}[r(y_1) - r(y)]dy. \end{aligned}$$

Note that $\int_{y_1}^{y_0} \frac{\bar{F}(y)}{\bar{F}(y_1)}[r(y_1) - r(y)]dy > 0$ since r is decreasing for $y < t_0$.
Now, we consider the second term given by

$$\int_{y_0}^{\infty} \frac{\bar{F}(y)}{\bar{F}(y_1)}[r(y_1) - r(y)]dy$$

$$= \int_{y_0}^{\infty} \frac{\bar{F}(y)}{\bar{F}(y_1)}[r(y_1) - r(y_0)]dy + \int_{y_0}^{\infty} \frac{\bar{F}(y)}{\bar{F}(y_1)}[r(y_0) - r(y)]dy$$

$$= \int_{y_0}^{\infty} \frac{\bar{F}(y)}{\bar{F}(y_1)}[r(y_1) - r(y_0)]dy + \frac{\bar{F}(y_0)}{\bar{F}(y_1)} \int_{y_0}^{\infty} \frac{\bar{F}(y)}{\bar{F}(y_0)}[r(y_0) - r(y)]dy.$$

The first term is +ve (since $r(y_1) > r(y_0)$) and the second term is $m'(y_0) = 0$.
Hence $m'(y_1) > 0$, which is a contradiction. Hence y_1 cannot be less than y_0.
Now, if $y_0 < y_1 < t_0$, by interchanging the roles of y_0 and y_1 in the above argument, we get $m'(y_0) > 0$, which is a contradiction meaning that y_0 cannot be less than y_1. Hence $y_1 = y_0$. Hence, there exists a unique $y_0 > 0$ such that $m'(y_0) = 0$ and that m attains a maximum at this point. Hence $m'(y) > 0$ for $y < y_0$ and $m'(y) < 0$ for $y > y_0$. Hence F is IDMRL.
Case (ii) : There does not exist a $y_0 > 0$ such that $m'(y_0) = 0$
Here either $m'(y) > 0$ for all $y > 0$ or $m'(y) < 0$ for all $y > 0$. From Lemma 2.1, it is clear that $m'(y) < 0$ for $y \geq t_0$. Therefore, $m'(y) < 0$ for all $t > 0$. Hence F is DMRL.
Remark 2.1 For the MRL function considered in (2.2), $m'(y) = -4.6y/[(1+2.3y^2)^2]$, hence case (ii) is satisfied implying that F is DMRL even though F is BFR.

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