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A SIMPLE INEQUALITY OF MOMENTS IN SOME CLASSES OF BIVARIATE AGEING DISTRIBUTIONS

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Abstract

We derive an inequality in terms of moments of Bivariate IFR distributions. We then show that this inequality holds good in a larger class of bivariate ageing distributions such as $NBU, NBUE$ and $DMRL$.

Key words and Phrases : *Memoryless property, Product Moments*

1 Introduction

In this paper, we consider the "very strong" versions of bivariate Ageing distributions defined by Buchnan and Singpurwalla (1977). The following classes are considered :

1. $BIFR$: A bivariate distribution $F(x,y)$ with $\overline{F}(0,0) = 1$ is said to have Bivariate Increasing Failure Rate ($BIFR$) distribution if

$$\overline{F}(x+u, y+v)/\overline{F}(x,y) \text{is non-increasing in } x, y \geq 0 \ \ \forall u, v \geq 0.$$

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2. $BNBU$: A bivariate distribution $F(x,y)$ with $\overline{F}(0,0)=1$ is said to have Bivariate New Better than Used ($BNBU$) distribution if
$$\overline{F}(x+u,y+v) \leq \overline{F}(x,y)\overline{F}(u,v) \ \ \forall \ \ x,y,u,v \geq 0.$$
3. $BNBUE$: A bivariate distribution $F(x,y)$ with $\overline{F}(0,0)=1$ is said to have a Bivariate New Better Used in Expectation ($BNBUE$) distribution if
$$\int_0^\infty \int_0^\infty \overline{F}(x+u,y+v)dxdy \leq \overline{F}(u,v)\int_0^\infty \int_0^\infty \overline{F}(x,y)dxdy \ \forall \ u,v \geq 0.$$
4. $BDMRL$: A bivariate distribution $F(x,y)$ with $\overline{F}(0,0)=1$ is said to have a Bivariate Decreasing Mean Residual Life (BDMRL) distribution if
$$\int_0^\infty \int_0^\infty \overline{F}(x+u,y+v)dxdy/\overline{F}(u,v) \text{ is non-increasing in } u,v \geq 0.$$

Here $\overline{F}(x,y)=1-F_X(x)-F_Y(y)+F(x,y)$ where $F_X(x),F_Y(y)$ are the marginal distributions of X and Y respectively. The dual classes of bivariate negative ageing distribution viz. $BDFR, BNWU,$

$BNWUE, BIMRL$ may be obtained by replacing "non-increasing" by "non-decreasing" and the inequality " $\leq$ " by the inequality " $\geq$ ". It can be easily seen that the following chain of implications hold good among the above classes of distributions :

$$\begin{array}{ccccc} F \text{ is } BIFR & \Longrightarrow & F \text{ is } BNBU & \Longrightarrow & F \text{ is BNBUE} \\ & \searrow & & \swarrow & \\ & & F \text{ is } BDMRL & & \end{array}$$

In this paper, we denote $\theta_i = E(X^iY^i), i \geq 1$, to represent the product moments of (X,Y). In Section 2 of this paper, we derive an inequality in terms of moments for $BIFR$ distributions, and in Section 3, we show that the same inequality holds good in the class of $BNBU, BNBUE$ and $BDMRL$ distributions also.

2 Inequality in the Class of $BIFR$ Distributions

Here, we derive a simple inequality in terms of the first two moments of the product XY for $BIFR$ distributions. First we state a useful lemma (see Kotz, Balakrishnan and Johnson (2000) p. 399-400 for a proof of the Lemma).

Lemma 2.1 : Let F be a bivariate distribution with $\overline{F}(0,0)=1$, which is continuous with respect to both the arguments. Then

$$\overline{F}(x+u,y+v)=\overline{F}(x,y)\,\overline{F}(u,v)\ \ \forall\ x,y,u,v\geq 0 \tag{1}$$

if and only if $\overline{F}(x,y)=e^{-\lambda x}e^{-\mu y}\ \ \forall\ x,y\ \geq 0$ for some $\lambda,\mu>0$.

Theorem 2.1 : Let F be a $BIFR$ distribution. Then $\theta_2\leq 4\theta_1^2$ with strict equality if and only if F is of the form $\overline{F}(x,y)=exp(-\lambda x-\mu y)$ for all $x,y\geq 0$ and for some $\lambda,\mu>0$.

Proof : Since F is $BIFR$, by taking $x_1=x>0, x_2=0, y_1=y>0, y_2=0$, we have

$$\overline{F}(x+u,y+v)\ \leq \overline{F}(x,y)\overline{F}(u,v)\ \ \forall u,v\ \geq\ 0,\ x,y,\ \geq 0. \tag{2}$$

For a bivariate distribution with $\overline{F}(0,0)=1$, we have (see Barlow and Proschan (1975), p. 135),

$$E(X^mY^n)=mn\int_0^\infty\int_0^\infty x^{m-1}y^{n-1}\overline{F}(x,y)dxdy \tag{3}$$

Consider

$$\theta_2=E(X^2Y^2)=4\int_0^\infty\int_0^\infty xy\overline{F}(x,y)dxdy$$

$$\begin{aligned}
&= 4\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \overline{F}(x,y)\ dudvdxdy\\
&= 4\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \overline{F}(x+u,y+v)\ dudvdxdy.
\end{aligned} \tag{4}$$

Using (2.2) and (2.4), we get

$$\theta_2 \leq \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \overline{F}(x,y)\ \overline{F}(u,v)\ dudvdxdy\ = 4E^2(XY) = 4\theta_1^2. \quad (5)$$

This proves the first part of the theorem.

Now, if $\overline{F}(x,y) = e^{-\lambda x - \mu y}$ for some $\lambda > 0, \mu > 0$, it can be easily seen that $\theta_2 = 4/\lambda^2\mu^2, \theta_1 = 1/\lambda\mu$ and hence $\theta_2 = 4\theta_1^2$ is attained. On the contrary, suppose that $\theta_2 = 4\theta_1^2$ for a BIFR distribution. This implies

$$4\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \ [\overline{F}(x+u, y+v) - \overline{F}(x,y)\ \overline{F}(u,v)]\ dudvdxdy = 0 \quad (6)$$

In view of (2.2), (2.6) will hold if and only if

$$\overline{F}(x+u, y+v) = \overline{F}(x,y)\overline{F}(u,v) \ \ \forall x,y,u,v\ \geq\ 0 \quad (7)$$

Proof of the second part of the Theorem now follows from Lemma 2.1.

3 Inequality of BNBU, BNBUE and BDMRL Classes

Theorem 3.1 : Let F be a Bivariate NBU distribution. Then $\theta_2 \leq 4\theta_1^2$ with the strict equality if $\overline{F}(x,y) = e^{-\lambda x - \mu y}$ for all $x, y \geq 0$ for some $\lambda, \mu > 0$.

Proof : Since F is $BNBU$, we have

$$\overline{F}(x+u, y+v) \leq \overline{F}(x,y)\overline{F}(u,v) \ \ \forall x,y,u,v\ \geq 0. \quad (1)$$

Note that (3.1) is the same as (2.2) of Theorem 2.1, which leads to the inequality $\theta_2 \leq 4\theta_1^2$ and the corresponding result concerning the strict equality.

Theorem 3.2: Let F be a $BNBUE$ distribution. Then $\theta_2 \leq 4\theta_1^2$.

Proof: Since F is $BNBUE$, we have

$$\int_0^\infty \int_0^\infty \overline{F}(x+u, y+v), dxdy \leq \overline{F}(u,v) \int_0^\infty \int_0^\infty \overline{F}(x,y)dxdy. \qquad (2)$$

Integrating both sides of (3.2), we get

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \overline{F}(x+u, y+v)\ dudvdxdy \leq \theta_1^2. \qquad (3)$$

Note that

$$\begin{aligned} & \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \overline{F}(x+u, y+v) dudvdxdy \\ = & \int_0^\infty \int_0^\infty \int_0^x \int_0^y \overline{F}(x,y) dudvdxdy \qquad (4) \\ = & \int_0^\infty \int_0^\infty xy\overline{F}(x,y)dxdy = \theta_2/4 \qquad (5) \end{aligned}$$

Hence (3.3) becomes $\theta_2 \leq 4\theta_1^2$ as desired in the theorem.

Note 3.1 : The inequality $\theta_2 \leq 4\theta_1^2$ holds good in $BDMRL$ class as it is a smaller class as compared to $BNBUE$.

Note 3.2 : It can be easily seen that the reverse inequality holds viz. $\theta_2 \geq 4\theta_1^2$ in the dual classes $BDFR, BNWU, BNWUE$ and $BIMRL$ distributions.

Note 3.3: It is well known that in the class of univariate IFR distributions $E(X^2) \leq 2(EX)^2$ and that equality holds if and only if F is exponential. This result has been used (see Doksum and Yandell (1984)) to derive a test for Exponentiality against IFR alternatives. Similarly using Theorem 2.1, in this paper, one may derive tests for $\overline{F}(x,y) = e^{-\lambda x - \mu y}$ against Bivariate IFR alternatives.

References

Barlow, R.E and **Proschan, F.** (1979). *Statistical Theory of Reliability and Life Testing.* Holt, Rinehart and Winston, Inc., New York.

Buchnan, W.B. and **Singpurwalla, N.D.** (1977). Some stochastic characterizations of Multivariate survival. In : *Theory and Applications of Reliability* (Editors : Tsokos C.P. and Shimi, I.P.) **1**, 329-348, Academic Press.

Doksum, K.A. and Yandell, B.S. (1984). Tests for Exponentiality. In : *Handbook of Statistics*, (Editors: Krishnaiah, P.R. and Sen, P.K.) **4**, 579-611.

Kotz, S., Balkrishnan, N. and Johnson, N.L. (2000). *Continuous Univariate Distributions : Models and Application.* **1**, Second Edition. John Wiley and Sons, New York.

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