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Testing exponentiality against L-distributions

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Abstract

In this paper we propose a test statistic for testing exponentiality versus L-class of life distributions. This test is based on an estimate of a functional of the cdf which discriminates between the exponential and L-family. © 1997 Elsevier Science B.V.

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1. Introduction and summary

We assume that F is an absolutely continuous life distribution with $F(0)=0$, has a finite mean μ and denote the survival function (SF) by $\bar{F}$ $(=1-F)$. Let D denote the set of all such life distributions.

Klefsjö (1983) has defined and extensively studied the L-class of life distributions. A distribution function $F \in D$ is said to belong to the L-class if for $s \geqslant 0$,

$$\int_0^\infty \exp(-st)\bar{F}(t)\,\mathrm{d}t \geqslant \mu(1+s\mu)^{-1}. \tag{1.1}$$

The right-hand side of (1.1) may be seen to be the Laplace transform of an exponential distribution with mean μ. It is easy to show that the L-class is strictly larger than the harmonically new better than used in expectation (HNBUE) class of life distributions. Hence, the L-class also contains the smaller NBUE, NBU, IFRA and IFR classes of life distributions.

An excellent survey of the tests of exponentiality versus the various life distributions belonging to IFR, IFRA, NBU, NBUE-classes is given in Hollander and Proschan (1984). The same problem for HNBUE-class was considered by Klefsjö (1983). For testing exponentiality against IDMRL distribution see Hawkins et al. (1992) and references therein.

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Let E denote the class of exponential distributions. Then $E \subset D$ and equality occurs in (1.1) for all $s \geqslant 0$ if and only if $F \in E$. Due to this "no-aging" property of $F \in E$, it is of practical interest to know whether a given life distribution F is in E. Alternatively, one may ask if $F \in L$.

Therefore, in this paper, we consider the problem of testing H_0: $F \in E$ versus H_1: $F \in L$ based on a random sample $X_1, \ldots, X_n$, of size n, from $F \in D$ (F unknown). Our test is based on an estimate of a functional which distinguishes $F \in E$ from $F \in L$. This functional for $F \in D$ is given by

$$\varphi(F) = \sup\{\psi_s(F)\colon\ 0 \leqslant s \leqslant F^{-1}(1-\varepsilon)\}, \tag{1.2}$$

where $1 > \varepsilon > 0$ is a small fixed number, and,

$$\psi_s(F) = \int_0^\infty \exp(-st)\bar{F}(t)\,\mathrm{d}t - \mu(1+s\mu)^{-1}. \tag{1.3}$$

The functional $\varphi(F)$ is clearly 0 for $F \in E$ and strictly positive for $F \in L$.

Our test statistic is $\varphi(F_n)$ where $F_n(\cdot)$ is the empirical df. We show that this statistic has a limit distribution which coincides with the distribution of the supremum of certain Gaussian process. Using this limiting distribution, critical values are obtained. We compute power by Monte-Carlo method.

The paper is organised as follows. In Section 2 the test statistic and its limiting null distribution are given. Section 3 contains the power computations.

2. The test statistic and its limiting null distribution

Let $\bar{X}_n$ denote the sample mean. Then our test statistic is

$$T_n = n^{1/2}\bar{X}_n^{-1}\varphi(F_n). \tag{2.1}$$

Lemma 2.1. *With definition* (1.3), *we have*

$$\sup_{0 \leqslant s \leqslant F^{-1}(1-\varepsilon)} \|\psi_s(F_n) - \psi_s(F)\| = O_p(n^{-1/2}), \tag{2.2}$$

where F_n is the empirical df and F is assumed to be a life distribution having an absolutely continuous density.

Proof. See the appendix.

Hence, by the above lemma, $\varphi(F_n)$ is near zero under H_0 and large positive under H_1, making large values of T_n significant for testing H_0 versus H_1.

The following computational formula is derived easily, where $X_{(1)} < \cdots < X_{(n)}$ denote the order statistics ($X_{(0)} = 0$):

$$T_n = n^{1/2}\bar{X}_n^{-1} \max_{1 \leqslant j \leqslant m} \left\{ \sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) a_{ij} - \frac{\bar{X}_n}{1 + (j/m)F^{-1}(1-\varepsilon)\bar{X}_n} \right\}, \tag{2.3}$$

where

$$m=[n(1-\varepsilon)],$$

and

$$a_{ij}=\frac{\exp(-(j/m)F^{-1}(1-\varepsilon)X_{(i)})-\exp(-(j/m)F^{-1}(1-\varepsilon)X_{(i+1)})}{(j/m)F^{-1}(1-\varepsilon)}.$$

The asymptotic null distribution of T_n is given in Theorem 2.1. In this direction, let

$$Z_\varepsilon(p)=\int_0^1(1-u)^{p\mu-1}W^0(u)\,\mathrm{d}u \quad \text{for some } \mu>0, \tag{2.4}$$

where W^0 is a Brownian bridge. Then $\{Z_\varepsilon(p),\ 0\leqslant p\leqslant 1-\varepsilon\}$ is a zero-mean Gaussian process with covariance function $K(p,q)$ given by

$$K^{-1}(p,q)=(p\mu+1)(q\mu+1)(p\mu+q\mu+1). \tag{2.5}$$

Theorem 2.1. *Under* H_0: $F\in E$,

$$T_n\xrightarrow{\omega}Z_\varepsilon\equiv\sup\{Z_\varepsilon(p)\colon\ 0\leqslant p\leqslant 1-\varepsilon\}.$$

Proof. See the appendix.

Approximate asymptotic critical values of the test statistic T_n based on Z_ε can be obtained via the following theorem.

Theorem 2.2. *For the Gaussian process* $\{Z_\varepsilon(p)\colon\ 0\leqslant p\leqslant 1-\varepsilon\}$ *defined in* (2.4),

$$P\left(\sup_{0\leqslant p\leqslant 1-\varepsilon}Z_\varepsilon(p)>c\right)\leqslant 2P\left(X>\frac{c}{K^{1/2}(1-\varepsilon,1-\varepsilon)}\right), \tag{2.6}$$

where X *is* $\mathrm{N}(0,1)$,

$$K^{-1}(1-\varepsilon,1-\varepsilon)=[\mu(1-\varepsilon)+1]^2[(1-\varepsilon)\mu+1] \text{ and } 0<\varepsilon<1.$$

Proof. See the appendix.

Table 1 contains selected quantiles of the distribution of Z_ε, computed from the bound in (2.6).

A comparison of the bound in (2.6) with the exact value is given in Table 3.

3. Power computation

Table 2 contains Monte-Carlo power computations based on 1000 replications of samples of size $n=50$ from $G(t)$, where

$$\bar{G}(t)=1-\frac{t}{\mu}\exp\left(1-\frac{t}{\mu}\right),\quad 0\leqslant t\leqslant\mu, \tag{3.1}$$

Table 1
Approximate quantiles of Z_ε ($\varepsilon = 0.10$ and $\mu = 1$)

α	0.90	0.95	0.99
Z_ε quantile	0.5189	0.6164	0.8114

Table 2
Power of the proposed test

Size μ	0.01	0.05	0.10
1	0.0430	0.0512	0.0758
5	0.2061	0.2105	0.2237
10	0.2397	0.2409	0.2816
20	0.4012	0.4653	0.4980
50	0.5574	0.5601	0.5645
100	0.6100	0.6205	0.6291

Table 3
A comparison of the bound in (2.6) with the exact value

Bound	Exact
0.10	0.07
0.05	0.03
0.01	0.008

see Chaudhuri and Deshpande (1996). It is known that $\bar{G}(t)$ is a logconcave SF, which is a lower bound of a SF belonging to the L-class. Note that the power increases as μ increases, as expected.

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Appendix

Proof of Lemma 2.1. We have,

$$\psi_s(F_n) - \psi_s(F) = \left(\int_0^\infty \exp(-st)\bar{F}_n(t)\,\mathrm{d}t - \frac{\mu}{1+s\mu} \right)$$

$$- \left(\int_0^\infty \exp(-st)\bar{F}(t)\,\mathrm{d}t - \frac{\mu}{1+s\mu} \right)$$
$$= \int_0^\infty \exp(-st)(\bar{F}_n(t) - \bar{F}(t))\,\mathrm{d}t$$
$$= -\int_0^\infty \exp(-st)(F_n(t) - F(t))\,\mathrm{d}t.$$

Thus,

$$\sup_{0\leqslant s\leqslant F^{-1}(1-\varepsilon)} n^{1/2}\ \|\psi_n(F_n) - \psi_s(F)\| \leqslant \int_0^\infty n^{1/2}\|F_n(t) - F(t)\|\,\mathrm{d}t$$
$$= \int_0^1 \|W_n(u)\| \frac{\mathrm{d}u}{f(F^{-1}(u))}.$$

(By applying the transformation $u = F(t)$ and writing $W_n(u) = n^{1/2}(F_n(F^{-1}(u)) - u)$, $\mathrm{d}u = f(F^{-1}(u))\,\mathrm{d}t$. Recall that F is assumed to have an absolutely continuous density f).

$$\xrightarrow{\omega} \int_0^1 \|W^0(u)\| \frac{\mathrm{d}u}{f(F^{-1}(u))} < \infty$$

(see Billingsley, 1968). This implies,

$$\int_0^1 \|W_n(u)\|\ \frac{\mathrm{d}u}{f(F^{-1}(u))} = \mathrm{O_p}(1).$$

This completes the proof of the lemma. □

Proof of Theorem 2.1. Consider the stochastic process

$$Z_n(p,F) = n^{1/2}(\psi_s(F_n) - \psi_s(F))$$
$$= -\int_0^\infty \exp(-s(p)t)n^{1/2}(F_n(t) - F(t))\,\mathrm{d}t,$$

where $s(p) = F^{-1}(p)$, $0 \leqslant p \leqslant 1-\varepsilon$.

Apply the transformation:

$$u = F(t)$$

and define

$$W_n = n^{1/2}(F_n(F^{-1}(u)) - u), \quad 0 \leqslant u \leqslant 1.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}u}F^{-1}(u) = \mu(1-u)^{-1},$$

we have,

$$\mu^{-1}Z_n(p,F) = -\int_0^1 (1-u)^{p\mu-1}W_n(u)\,\mathrm{d}u.$$

Note that $\psi_s(F) = 0$, for F being exponential with mean μ.

Hence,

$$n^{1/2}\psi_s(F_n)\mu^{-1} \xrightarrow{\omega} -\int_0^1 (1-u)^{p\mu-1}W^0(u)\,\mathrm{d}u,$$

since $W_n(u) \xrightarrow{\omega} W^0(u)$ in $D[0,1]$, where $W^0(u)$ is a Brownian bridge. □

Remark. The case when $p = 0$ can be treated separately.

Proof of Theorem 2.2. Since $\{Z_\varepsilon(p),\ 0 \leqslant p \leqslant 1-\varepsilon\}$ has a continuous sample path with probability one, we have an a.s. Karhumen–Loeve expansion for $Z_\varepsilon(p)$.

$$Z_\varepsilon(p) = \sum_{j=1}^{\infty} z_j\phi_j(p),$$

where

$$z_j = \int_0^{1-\varepsilon} Z_\varepsilon(p)\phi_j(p)\,\mathrm{d}p$$

and ϕ_j is an eigenfunction corresponding to the eigenvalue λ_j such that

$$\int_0^{1-\varepsilon} K(p,q)\phi_j(p)\,\mathrm{d}p = \lambda_j\phi_j(q).$$

Here z_j's are independent $\mathrm{N}(0,\lambda_j)$, $j = 1,\ldots,\infty$. Consider

$$Z_\varepsilon^{(m)}(p) = \sum_{j=1}^{m} z_j\phi_j(p).$$

Then

$$P\left(\sup_{0\leqslant p\leqslant 1-\varepsilon} Z_\varepsilon^{(m)}(p) > c\right) \leqslant 2P(Z_\varepsilon^{(m)}(1-\varepsilon) > c),$$

since

$$P\left(\max_{1\leqslant j\leqslant n} Z_\varepsilon^{(m)}\left(j\frac{1-\varepsilon}{n}\right) > c\right) \leqslant 2P(Z_\varepsilon^{(m)}(p) > c)$$

(see Ash and Gardner, 1975, p. 180) and as $n \to \infty$,

$$P\left(\max_{1\leqslant j\leqslant n} Z_\varepsilon^{(m)}\left(j\frac{1-\varepsilon}{n}\right) > c\right) \to P\left(\sup_{0\leqslant p\leqslant 1-\varepsilon} Z_\varepsilon^{(m)}(p) > c\right),$$

which is due to the continuity of the sample path of $\{Z_\varepsilon(p),\ 0 \leqslant p \leqslant 1-\varepsilon\}$.

Now, since

$$Z_\varepsilon^{(m)}(p) \to Z_\varepsilon(p), \text{ uniformly in } p \quad \text{as } m \to \infty$$

(see Adler, 1990),

$$\sup_{0 \leqslant p \leqslant 1-\varepsilon} Z_\varepsilon^{(m)}(p) \to \sup_{0 \leqslant p \leqslant 1-\varepsilon} Z_\varepsilon(p) \quad \text{as } m \to \infty.$$

Hence,

$$P\left(\sup_{0 \leqslant p \leqslant 1-\varepsilon} Z_\varepsilon(p) > c\right) \leqslant 2P(Z_\varepsilon(1-\varepsilon) > c).$$

This completes the proof. □

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