Distribution-free CUSUM Control Chart

for Joint Monitoring of Location and Scale

S. Chowdhury
Indian Institute of Management
Kozhikode; QM & OM Area;
Kozhikode; Kerala,India.
Email: shovanc@iimk.ac.in

A. Mukherjee
Indian Institute of Management
Udaipur; OM, QM& IS Area,
Udaipur, Rajasthan-313001, India.

S. Chakraborti
Department of Information Systems,
Statistics and Management Science,
University of AlabamaTuscaloosa, USA

Abstract: Mukherjee and Chakraborti (2012) proposed a single distribution-free (nonparametric) Shewhart-type chart based on Lepage (1971) test statistic for simultaneously monitoring both the location and the scale parameters of a continuous distribution when both of these parameters are unknown. In the present work, we consider a single distribution-free CUSUM chart based on Lepage statistic, referred to as CUSUM-Lepage (CL) chart. Our proposed chart is nonparametric and therefore, in control (denoted IC) properties of the chart remain invariant and known for all continuous distributions. Control limits are tabulated for implementation in practice. The IC and out of control (denoted OOC) performance properties of the chart are investigated through simulation studies in terms of the average, the standard deviation, the median and some percentiles of the run length distribution. Detailed comparison with the Shewhart-type chart is presented. We also examine the effect of the reference value (k) of CUSUM chart on the performance of CL chart. The proposed chart is illustrated through a real data. Summary and conclusions are presented.

Keywords: Ansari-Bradley Statistic; Average Run Length; Phase I and II; CUSUM Lepage Chart; Nonparametric; Monte-Carlo simulation; Statistical process control; Shewhart Lepage Chart; Wilcoxon Rank sum Statistic.

1. INTRODUCTION

Robustness of many of the available control charts for monitoring a process largely depends on the assumption of normality which is often difficult to justify in practice. In the recent times, a number of researchers have advocated using distribution-free (nonparametric) control charts, in particular when the process distribution is unknown, or known to be markedly

different from normal or a heavy-tailed one. Nevertheless, most of the distribution-free control charts that are available in the literature are designed for monitoring either the process location or the scale parameter separately. Using separate charts for different process parameters can cause practical problems with regard to implementation and interpretation. Thus a single chart (as opposed to two separate charts) for joint monitoring of location and scale parameters has been recommended as it may be simpler and may have some performance advantages (see for example, McCraken et al., 2013). To understand the importance and impact of joint monitoring, readers may see the review by Cheng and Thaga (2006) for coverage of literature until 2005 and the paper by McCracken and Chakraborti (2013) for more recent advances.

The CUSUM charts were first introduced by Page (1954). Over the years, CUSUM charts have proven (see, for example, Hawkins (1987), Woodall (1983), Lucas (1985), Chang and Gan (1995)) to be useful in SPC and many other areas for monitoring processes over time. Various modifications and adaptations of CUSUM charts have also been proposed in the literature (see, for example, Reynolds et al. (1990), Gan (1993), Goel (2011) and Mukherjee et al. (2013)). The CUSUM chart is known to be superior to the Shewhart control chart in the sense that the CUSUM control charts tend to have smaller Average Run Lengths (ARL's) particularly for small changes in the parameters. While a Shewhart chart is better in detecting an immediate abrupt (transient) change, the cumulative sum (CUSUM) chart is more effective in detecting more sustained changes. The reader is referred to Hawkins and Olwell (1998) and Gan (2007) for a detailed discussion on CUSUM control charting literature.

While much work has been done and continues to be done in the parametric setting, it is now well recognized that the distribution-free (or nonparametric) charts are useful and expected to be superior when the model assumptions such as normality is difficult to validate. There has been a significant amount of work done in this area. While Park and Reynolds (1987) developed nonparametric procedures for monitoring location parameter of a continuous process based on linear placement statistic, McDonald (1990) considered a CUSUM procedure for individual observations based on the sequential ranks statistic. Bakir and Reynolds (1979) and Amin et al (1995) proposed a non parametric CUSUM chart based on signed-rank and sign statistics, respectively. Run-length distribution of the CUSUM chart was discussed in detail by Jones et al. (2004). Li et al (2010) considered the Wilcoxon rank sum test to detect step mean shifts through CUSUM and EWMA charts. Recently, Yang and Cheng (2011) and Mukherjee et al. (2013) developed nonparametric CUSUM charts to detect the possible small shifts in process mean. Recently, Ross et al. (2011) discussed nonparametric monitoring of data streams for changes in location and scale and Ross and Adams (2012) considered two nonparametric control charts for detecting arbitrary distribution changes. For an overview of nonparametric control charts, see Chakraborti et al. (2001, 2007, 2011).

For monitoring both the location and scale parameters in Phase II using a reference sample from Phase I, Mukherjee and Chakraborti (2012) considered a nonparametric Shewhart-Lepage (SL) chart based on the Lepage (1971) statistic. For the same problem, Chowdhury et al. (2013) considered a nonparametric Shewhart-Cucconi (SC) chart based on the Cucconi (1968) test statistic. Encouraged by these findings, in this paper, we take the work a step further in a new direction and consider a nonparametric CUSUM chart based on the Lepage (1971) statistic. The proposed CUSUM-Lepage (CL) charts are expected to be more effective in detecting smaller, more sustained types of shifts.

The rest of the paper is organized as follows. Section 2 provides a brief background of nonparametric control charting procedures for joint monitoring of location and scale parameters on the basis of the Lepage statistic, along with the statistical framework and preliminaries. The proposed CL control chart is introduced in Section 3. Section 4 is devoted to the derivation of the run length distribution, determination of the upper control limit (UCL) and examining the IC performance of the chart. The OOC performance of the CL chart, along with a detailed comparison with the SL chart, is presented in Section 5 based on various run length distribution characteristics obtained via Monte-Carlo simulation. In Section 6, we study the effect of the key parameter of CUSUM chart, the so-called reference value, on the performance of the chart. The charting procedure is illustrated in Section 7 with a data set from Montgomery (2005). We conclude with a summary in Section 8.

2. STATISTICAL FRAMEWORK AND PRELIMINARIES

Simultaneous monitoring of location and scale parameters is useful in many applications. A nonparametric control chart is useful when assumptions related to the process distribution cannot be made. While most of the distribution-free control charts have been devoted to monitoring the location parameter only, Mukherjee and Chakraborti (2012) and Chowdhury et al. (2013) proposed nonparametric control charts to monitor both the location and scale parameters. The Lepage (1971) test is a combination of the Wilcoxon rank sum test (for location) and the Ansari-Bradley test (for scale), and is used to test for the equality of location and scale parameters in the nonparametric literature. Mukherjee and Chakraborti (2012) used the Lepage statistic in a Shewhart type control chart to monitor the location and the scale parameters. The same Lepage statistic is used here in a CUSUM chart and the resulting chart is called the CUSUM-Lepage (CL) chart.

Let $U_1, U_{2,\dots,} U_m$ and $V_1, V_2, \dots, V_n$ be the independent random samples from two populations with continuous distribution functions (cdf) $F(x)$ and $G(y) = F\left(\frac{x-\theta}{\delta}\right)$; $\theta \in \Re$; $\delta > 0$; where F is an unknown continuous cdf. The constants θ and δ represent the unknown location and scale parameters, respectively. Introduce an indicator variable $I_k = 0$ or 1 as the k-th order statistic of the combined sample of $N\ (= m + n)$ observations is a U or a V, respectively. Further, consider the Wilcoxon rank sum (WRS) statistic, say, $T_1 = \sum_{k=1}^{N} kI_k$ used to test $\theta = 0$. Similarly, consider the Ansari-Bradley (AB) test statistic, say, $T_2 = \sum_{k=1}^{N} \left|k - \frac{N+1}{2}\right| I_k$, which is a popular choice for testing $\delta = 1$. It is known that the WRS and the AB statistics are mutually independent under IC and both the WRS and the AB tests are powerful for a class of distributions. Like, WRS test is most powerful when underlying population distribution is logistic. Interested readers may see Gibbons and Chakraborti (2010) for further details.

In the process monitoring context, let the U's denote the Phase I reference data and let the V's denote the Phase II (test) data under monitoring. In Phase II, the process is said to be in control (IC) when $F = G$, that is when $\theta = 0$ and $\delta = 1$. It is well known (see, Gibbons and Chakraborti (2010)) that $E(T_1|IC) = \mu_1 = \frac{1}{2}n(N+1)$ and $Var(T_1|IC) = \sigma_1^2 = \frac{1}{12}mn(N+1)$. Moreover,

$$E(T_2|IC) = \mu_2 = \begin{cases} \dfrac{nN}{4} & if \quad N\ is\ even \\ \dfrac{n(N^2-1)}{4N} & if \quad N\ is\ odd \end{cases}$$

and

$$Var(T_2|IC) = \sigma_2^2 = \begin{cases} \dfrac{1}{48}mn\dfrac{(N^2-4)}{N-1} & if \quad N\ is\ even \\ \dfrac{1}{48}\dfrac{mn(N+1)(N^2+3)}{N^2} & if \quad N\ is\ odd \end{cases}$$

Writing the standardized WRS and the AB statistics as $S_1 = \frac{T_1 - \mu_1}{\sigma_1}$ and $S_2 = \frac{T_2 - \mu_2}{\sigma_2}$, repectively, we may define the Lepage test statistic as: $S_L^2 = S_1^2 + S_2^2$. It is easy to see that $E(S_1|IC) = E(S_2|IC) = 0$ and $E(S_1^2|IC) = E(S_2^2|IC) = 1$ and therefore, $E(S_L^2|IC) = 2 = \mu_L$, say. Note that, S_L^2 is non negative by definition. Also, it is easy to visualize that whenever θ deviates from 0, irrespective of direction of the shift, the absolute value (ignoring sign) of S_1 is expected to be larger on an average. Similarly, irrespective of the direction of the shift of δ from 1, S_2 is expected to be larger on an average. As a consequence, irrespective of the type and the direction of the shift, either in location or in scale or in both, S_L^2 is expected to take on larger values compared to μ_L. Hence, it is clear that one should only focus at detecting one sided (rightward) shift in the Lepage statistic to identify any shift(s) in location and/or scale parameters. Thus, in order to monitor small change(s) in location and/or scale parameter, we propose an upper one sided CUSUM chart based on the Lepage statistic.

3. CONSTRUCTION OF CUSUM-LEPAGE (CL) CHART

The upper one-sided CUSUM chart based on Lepage Statistic, referred to as the CL chart may be constructed as follows:

Step 1. Collect a reference sample $\boldsymbol{X_m} = (X_1, X_2, \dots, X_m)$ of size m from an IC process. Establishing a reference sample is itself a challenging problem and there are several Phase I control charts available in the literature for this purpose. In this paper, emphasis is on evaluating the performance of the CL chart and we are assuming that an appropriate reference sample is available a-priori.

Step 2. Collect $\boldsymbol{Y_{j,n}} = (Y_{j1}, Y_{j2}, \dots, Y_{jn})$, the j-th Phase II (test) sample of size n, $j = 1, 2, \dots$

Step 3. Identify the *U*'s with the *X*'s and the *V*'s with the *Y* 's respectively. Calculate the WRS statistic T_{1j} and the AB statistic T_{2j} using the reference sample and the *j*-th test sample and obtain the standardized WRS and AB statistics S_{1j} and S_{2j}, respectively as described in Section 2. Finally obtain the Lepage statistic S_{Lj}^2 for the *j*-th test sample.

Step 4. Recall that $E(S_L^2|IC) = 2$ and therefore the CL plotting statistic is given by:

$C_j = max\left[0 ,\ C_{j-1} + \left(S_{Lj}^2 - 2\right) - k\right]; j = 1, 2, ...,$ for the *j*-th test sample with the starting value $C_0 = 0$. Here, $k(\geq 0)$ is called a reference value.

Step 5. Plot C_j against an upper control limit (UCL) *H*. The lower control limit (LCL) is 0 by definition for this one sided CUSUM chart.

Step 7. If C_j exceeds *H*, the process is declared OOC at the *j*-th test sample. If not, the process is thought to be IC and monitoring continues to the next test sample.

Step 8. Follow-up: Recently Chowdhury et al. (2013) and McCracken et al. (2013) introduced the idea of *p*-value based follow up when a process is declared OOC. We follow the same idea and compute the *p*-values for the Wilcoxon test for location and the Ansari-Bradley test for scale respectively, based on the two samples; one with the *m* Phase I observations, and the other with the *n* observations from the *j*-th test sample. Denote the *p*-values of the corresponding tests as p_1 and p_2 respectively. As in Chowdhury et al. (2013) and McCracken et al. (2013) we argue that if p_1 is very low but not p_2, a shift in only location is indicated. If p_1 is relatively high but p_2 is low; only a shift in scale is suspected. If both *p* values are very low; a shift in both location and scale is declared.

4. RUN LENGTH DISTRIBUTION

Brook and Evans (1972) and Woodall (1984) among others proposed approximating the run length distribution of various CUSUM procedures when parameters are known by using the notion of a Markov process. The run length distribution of a Phase II CUSUM process may be approximated by a Markov Process given the reference sample $\boldsymbol{X_m}$, and the conditional average run length may be obtained using the properties of the Markov process. Since the conditional average run length is a random variable, one can find and use the unconditional average run length by integrating (averaging) over all possible conditional run lengths.

In the present context, however, the exact conditional distribution of the Lepage statistic is itself complicated with no clear explicit form. As a consequence, the exact unconditional run length distribution of the proposed CUSUM Lepage procedure appears intractable. However, interested readers may explore the possibility of deriving a suitable approximation, noting that, given $\boldsymbol{X_m}$, and as both m, n tend to ∞ such that $\frac{m}{m+n} \to \lambda,\ 0 < \lambda < 1,\ S_i^2 = S_{1i}^2 + S_{2i}^2$ follows a generalized chi-square distribution (see Jones (1983)) with non-centrality parameters:

mean vector: $\left(\frac{\mu_1(\boldsymbol{X_m})-\mu_1}{\sigma_1}, \frac{\mu_2(\boldsymbol{X_m})-\mu_2}{\sigma_2}\right)'$, variance-covariance matrix: $\begin{pmatrix} \frac{\sigma_1^2(\boldsymbol{X_m})}{\sigma_1^2} & \frac{\sigma_{12}(\boldsymbol{X_m})}{\sigma_1\sigma_2} \\ \frac{\sigma_{12}(\boldsymbol{X_m})}{\sigma_1\sigma_2} & \frac{\sigma_2^2(\boldsymbol{X_m})}{\sigma_2^2} \end{pmatrix}$**,** where

$$\mu_1(\boldsymbol{X_m}) = \frac{n(n+1)}{2} + n\sum_{i=1}^{m}\left[\bar{G}(X_i) - \frac{1}{2}\right], \mu_2(\boldsymbol{X_m}) = \sum_{i=0}^{n}\sum_{j=1}^{m}\left|\frac{m+n+1}{2} - j - i\right|\binom{n}{i}\left[G\left(X_{(j)}\right)\right]^i\left[\bar{G}\left(X_{(j)}\right)\right]^{n-i},$$

$$\sigma_1^2(\boldsymbol{X_m}) = n\left\{\sum_{i=1}^{m}\bar{G}(X_i)G(X_i) + \sum_{i=1}^{m}\sum_{j=1(\neq i)}^{m}\left[\bar{G}\left(X_i \wedge X_j\right) - \bar{G}(X_i)G\left(X_j\right)\right]\right\},$$

with $\bar{G} = 1 - G$ and $X_i \wedge X_j = max\left(X_i,\ X_j\right)$,

$$\sigma_2^2(\boldsymbol{X_m}) = \sum_{i=0}^{n}\sum_{j=1}^{m}\left(\frac{m+n+1}{2} - j - i\right)^2\binom{n}{i}\left[G\left(X_{(j)}\right)\right]^i\left[\bar{G}\left(X_{(j)}\right)\right]^{n-i} - \mu^2{}_2(\boldsymbol{X_m}),$$

and

$$\sigma_{12}(\boldsymbol{X_m}) + \mu_1(\boldsymbol{X_m})\mu_2(\boldsymbol{X_m}) = n\sum_{i=0}^{n}\sum_{j=1}^{m}(n-i)\left|\frac{m+n+1}{2} - j - i\right|\binom{n}{i}\left[G(X_{(j)})\right]^{i}\left[\bar{G}(X_{(j)})\right]^{n-i}$$

$$+\frac{n}{2}\sum_{i=0}^{n-1}\sum_{i'=i}^{n}\sum_{j=1}^{m-1}\sum_{j'=j+1}^{m}\frac{n!\,(n-i)\left|m+n+1-2(j'+i')\right|}{i!\,(i'-i)!\,(n-i')!}\times\left[G(X_{(j)})\right]^{i}\left[G\left(X_{(j')}\right)-G(X_{(j)})\right]^{i'-i}\left[\bar{G}\left(X_{(j')}\right)\right]^{n-i'}$$

$$+\frac{n}{2}\sum_{i=0}^{n}\sum_{i'=0}^{i}\sum_{j=2}^{m}\sum_{j'=1}^{j-1}\frac{n!\,(n-i)\left|m+n+1-2(j'+i')\right|}{i'!\,(i-i')!\,(n-i)!}\times\left[G\left(X_{(j')}\right)\right]^{i'}\left[G(X_{(j)})-G\left(X_{(j')}\right)\right]^{i'-i}\left[\bar{G}(X_{(j)})\right]^{n-i}$$

An outline of the proof is given in the Appendix. For computing approximate conditional ARL under the IC set up via Markov chain approach, one needs to obtain several transition probabilities. Those may be computed with some effort using the above approximate distribution. However, transition probabilities do not have simple explicit forms. Further approximations to obtain the asymptotic unconditional distribution appears very complicated and will be a major comutational challenge to meet in future.

Given these complexities in the derivations of the conditional and the unconditional average run lengths, we employ Monte-Carlo simulations to evaluate the necessary quantities. Details of the simulations and the results are discussed in subsequent sections and subsections.

4.1. Determination of *H*

Numerical computations in R.2.14.1 software, based on Monte-Carlo simulations are used to determine *H* on the basis of 50,000 replicates. Because of the distribution-free nature of the CL chart, we generate *m* observations from a standard normal distribution for the Phase I sample and *n* observations from the same distribution for each test sample. The results, which are displayed in Table 1, show a pretty stable and meaningful estimates of the IC average run length (ARL_0) and other percentiles of the IC run length distribution. We have chosen *m* = 30, 50, 100 and 150 for the reference sample size and *n* = 5 and 11 for the test sample size as in Mukherjee and Chakraborti (2011). The values of k, one of the key charting parameters are chosen as 0, 3 and 6. Note that since $Var(S_i^2|IC) = 4,$ we consider 1.5 and 3 times the standard deviation of plotting statistic as the possible choices of k to cover different situations. No substaintial changes in the ARL figures could be obtained with minor (less than 0.5 times standard deviation of plotting statistic) changes in k and moreover, H and ARL values for any k in (0,6) can be obtained via interpolation using the three values of k as used in this article. For any given triplet (m,n,k), a search is conducted to obtain the appropriate H value that ensures the ARL_0 is close to a nominal (target) value. The fourth to the sixth columns of Table 1 give the required H values for target ARL_0 = 250, 370 and 500 respectively. Thus, for example, when 30 reference observations and test samples of size 5 are available with k = 3 and an ARL_0 of 500 is desired, the upper control limit H for the CL chart is given by 4.617. To justify the nonparametric nature of the proposed chart, it can be easily verified that for a given combination of ($m,n,k;ARL_0$) the same H values as in Table 1 are valid for any other non-normal distribution because of the distribution-free nature of the plotting statistic under IC set-up.

We see from Table 1 that that for any fixed combination of (m,n,k) values, the higher the nominal ARL_0 values, the higher the values of H. Further for fixed n and k, H increases with the increase in the reference sample size m but the H decreases with an increase in the test sample size n for fixed m and k in almost all the cases except for a very few sampling fluctuations. Finally, as m and n both increase, H values tend to stabilizes as a function of the ratio $\lambda = \frac{n}{m+n}$.

<Table-1 Here>

4.2. IC performance of the Chart

<Table-2 Here>

Table 2 shows that the IC run length distribution is highly right skewed and consequently is worthwhile to study various summary measures viz. the mean, the SD and several percentiles including the first and the third quartiles. As mentioned earlier in 4.1, we simulate both the reference and the test samples from the standard normal distribution for m = 30, 50, 100, 150 and n = 5, 11 and k = 0, 3, and 6. For a given triplet (m, n, k), we obtain H from Table 1, for a nominal (target) ARL_0 of 500 and simulate various characteristics of the IC run length distribution.

It is observed that the target ARL_0 value 500 is much higher than the medians for all (m,n,k) combinations. When m = 100 or 150, the median is more or less half of the target ARL_0 = 500. As the reference sample size m increases from 30 to 150, for a fixed n and k, all the percentiles including the median increase except the 95th percentile and the SD decreases. The 95th percentile is seen to be more or less stable around 3.6 to 4.4 times the target ARL_0 = 500 except for the cases when m = 30 and 50 with k = 0. It is also observed that for fixed m and k, as test sample size increases, all percentiles except 3rd quartile and 95th percentile decrease and SD increases. It is also of worth mentioning that in the majority of the combinations of (m,n,k), the 3rd quartiles of the run length distributions are closer to the nominal IC ARL_0 value of 500. This indicates that the IC run length distribution of the proposed chart, like the same for many other control charts, is heavily skewed with a long right tail.

5. PERFORMANCE COMPARISONS

We consider two popular distributions under the general location-scale family in order to facilitate the OOC performance comparisons. First (case I), the thin tailed symmetric normal distribution (N (θ, δ)) with pdf $f(u) = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2\delta^2}(u-\theta)^2}, u \in (-\infty, \infty),$ and second (case II), the heavy tailed symmetric Cauchy distribution (Cauchy (θ; δ)) with pdf $f(u) = \frac{\delta}{\pi(\delta^2 + (u-\theta)^2)}$,

$u \in (-\infty, \infty)$. We examine the performance characteristics of the run length distribution when the IC sample in each case is taken from the corresponding standard distribution with θ = 0 and δ = 1. Thus in case I, the IC samples are taken from a N(0,1) distribution, with the OOC sample coming from a N(θ,δ) distribution. To examine the effects of shifts in the mean and the variance, 40 combinations of (θ, δ) values are considered viz. θ = 0, 0.25, 0.5, 0.75, 1, 1.5, 2 and 3 and δ = 1, 1.25, 1.5, 1.75 and 2 respectively. We not only study the performance of the CL charts for these two distributions, but also compare with that for the SL chart of Mukherjee and Chakraborti (2012) for the same combinations of shifts. For brevity, only the results for m = 50, 100 and n = 5 are presented in Table 3. In case II, we examine the chart performance characteristics for the heavy tailed and symmetric Cauchy distribution for both the CL and SL charts, using the same combinations of the reference and test sample sizes and location and scale parameters (θ and δ), with the IC sample coming from a Cauchy(0,1) distribution. These results are shown in Table 4. Note that in Tables 3 and 4, the first row of each of the cells shows the ARL and (SDRL) values, whereas the second row shows the 5^{th}, 25^{th}, 50^{th}, 75^{th}, 95^{th} percentiles (in this order).

In general, the simulation results reveal that the OOC run length distributions are also skewed to the right and this can be observed from Tables 3 and 4 for both charts. Moreover, except for minor sampling fluctuations, for fixed m, n, k and a given ARL_0, the OOC ARL values as well as the percentiles all decrease sharply with the increasing shift in the location and also with the increasing shift in the scale. This (expected) phenomenon is seen for both the CL and SL charts and this indicates that both distribution-free charts are reasonably effective in detecting shifts in the location and/or on the scale. However, the effectiveness of the chart (speed of detection) varies depending on the type of shift and the type of chart being considered. Both CL and SL charts detect a shift in the scale faster than that in the location.

For example, from Table 3, we see that for a 25% increase in the location parameter (θ) when the scale parameter (δ) is in IC, there is about a 41% reduction in the ARL for the SL chart and about 38-51% reduction in the CL chart with *m* = 50 and varying choices of *k*. Moreover, in the same situation, for *m* = 100, we find nearly a 49% reduction in the ARL of the SL chart and about 50-63% reduction in that of the CL chart. However, for a 25% increase in the scale parameter when the location parameter is in IC, there is about 78.7% and 80% reduction in the ARL for the SL chart with both *m* = 50 and 100 respectively. On the other hand, we observe a reduction of 79-87% in the ARL of the CL chart for *m* = 50 and nearly 81-89% reduction for *m* = 100 and different choices of *k*. Finally, when both the location and scale parameters increase by 25%, the ARL for the SL chart decreases by nearly 85% and the same for the CL chart reduces by nearly 85-91% for *m* = 50.

The pattern is quite similar for the SDRL for both charts. For example, we see that for a 25% increase in the location when the scale is IC, the SDRL decreases for an increase in the shift in both parameters but decreases at a faster rate for a shift in the scale parameter. For example, when *m* = 100, for a 25% increase in the location only, the SDRL decreases by 44.5% for the SL chart and around 38-51% for the CL chart, while for a 25% increase in the scale parameter only, the SDRL decreases by 83.2% for the SL chart and 82-93% for the CL chart. From Table 3, it is interesting to note that the CL chart with suitable *k*, performs better than the SL chart not only in detecting small shifts in the parameters, but for larger shift as well.

Next, it is useful to examine their OOC performance for underlying distributions that have tails heavier than the normal since heavier-tailed symmetric distributions under the location-scale family, such as the Cauchy distribution arise in applications where extreme values can occur with higher probability. Keeping this in mind we repeat the simulation study with data from the Cauchy distribution. The performance characteristics of the run length distribution were evaluated when the IC sample is taken from a Cauchy(0,1) distribution, but the test samples are from a Cauchy(θ,δ) distribution. To study the impact of a shift in the location and scale, as in the normal case, we study the same 40 combinations of θ and δ values. From Table 4, it is seen that for the Cauchy distribution, the general patterns in the OOC ARL values remain the same as in the case of the normal distribution, but the magnitudes of the ARL values are much higher for a similar shift in the location or in the scale parameter, indicating a moderately slower detection of shifts under the heavier-tailed distribution. For example, when m = 100, k = 0 and the location and scale both increase by 25%, the ARL is 161.7 compared to 39.3 in the normal case for the CL chart. Moreover, the percentiles as well as the SDRL values all increase under the Cauchy distribution.

In summary, Tables 3 and 4 show that for any kind of shift in the process average or/and variability, the proposed CL chart outperforms the SL chart. However, the variation in the performance of the CL chart can be further explained by the reference value, k which is discussed in the next section. In this context, it is worth mentioning that to the best of our knowledge, there is no competing parametric Phase II CUSUM chart available in the literature for joint monitoring of parameters of a normally distributed processes. However, if we compare the normal theory Modified Max Chart or the Modified Shewhart Distance chart as in McCraken et al. (2013) with the nonparametric SL chart, we see that the non parametric chart for joint monitoring is as good as its parametric counterpart or better in many situations. The reason is that the distance or the max type charts are Shewhart type charts based on the combination of two optimal test statistics but not on a single optimal test. Robust performance of Lepage Statistic in joint monitoring over the Max or the distance type chats also motivates us to develop a CUSUM chart based on Lepage statistic.

6. EFFECT OF *k* ON THE PERFORMANCE OF THE CHART

It is seen that the reference value, k, of the CL chart has a significant impact on both the IC and OOC performance of the chart. In the IC set up, for fixed m and n and ARL_0, we can see from Table 1 that the value of H decreases with the increase in k. Table 2 shows that $SDRL_0$ is relatively higher when $k = 0$ but stabilizes as k increases.

In the OOC set-up, it is evident from Table 3 that the CL chart is able to detect smaller shifts in location quite successfully with $k = 0$, uniformly for all choices of m and n under the normal distribution when variability is under control, which is quite expected for a CUSUM chart. But, interestingly, the CL chart out performs the SL chart, even for moderate to large shifts in the location (that is, $\theta \geq 1$) with $k = 3$ or 6, except for $\theta = 3$, where both the charts perform similarly. The CL chart displays nearly similar behavior for a scale shift when underlying model is normal with a stable process location. We see that if $\theta = 0$, the CL chart with $k = 0$ is the best for $1 < \delta \leq 2$. From Table 3, we further see that for small shifts in location along with small to large shifts in scale, the CL chart shows best performance for $k = 0$ and for large shifts in location accompanied by small to large shifts in scale, the same chart performs best for $k = 3$. Note that the OOC performance of the CL chart with $k = 6$ shows nearly similar OOC performance as that of the SL chart and both are usually the best choice to detect large shift in mean.

OOC performance of the CL chart for the Cauchy distribution has both similarity and dissimilarity with the same for the normal distribution. Table 4 shows that for $m = 50$, the CL chart performs better than the SL chart in detecting small, moderate to large shifts in location or scale or both for only k = 0. There is only one unusual behaviour, we have observed for the case of $\theta = 0$ and $\delta = 1.25$, where k = 6 gives the best result for the CL chart. For m = 100, the same characteristics are shown by the CL chart for k = 0 except in one case where there is a large shift in location irrespective of shifts in scale and where the CL chart of course performs better for k = 3. Table 4 also reveals that for k = 0, ARL in the CL chart decreases by 17% for m = 50 and by 21% for m = 100 as compared to a very small reduction in ARL in SL chart for a 25% increase in the location.

In general, it is observed that the CL chart performs better in terms of both ARL and SDRL for k = 0 for both the normal and Cauchy distributions for shifts in location, scale or both. There are a few exceptions to this in both the distributions as stated above. It is also noted that the magnitude of reduction in ARL or SDRL is much higher in case of normal distribution than Cauchy for different types of shits in process parameters. In general, it is recommended to consider smaller values of k, say k = 0 for detecting smaller to medium shift in thin tailed distributions and smaller to larger shifts in the case of heavy tailed distributions. If the target shift is relatively large in a thin tailed distribution or very large in a heavy tailed distribution, one should use k = 3 to get a quick OOC signal.

7. ILLUSTRATIVE EXAMPLE

Here, we illustrate the proposed nonparametric CLchart using the well-known piston ring data in Montgomery (2005) (Table 5.1 and 5.2, respectively). Piston rings for an automotive engine are produced by a forging process. The goal is to establish statistical control of the inside diameters of the rings manufactured by this process. Twenty five samples each of size 5, shown in Table 5.1 of Montgomery (2005), are taken. A Phase I analysis in Montgomery (2005) concluded that we may consider this data set with 125 observations, as the set of reference data. Further, in Table 5.2 of Montgomery (2005), fifteen Phase II samples (test samples) each of size 5 are given. That is, for the present purpose, n = 5. Using simulations, for a target ARL_0 of 500, for m = 125, n = 5 and for k = 0, 3 and 6, we find H to be 28.08927, 6.804037 and 3.445849 respectively. The lower control limit is 0 by default.The fifteen cusum-Lepage plotting statistics are given in Table 5 and shown in the following figures for the three choices of k.

Table 5: CL Plotting statistics for m = 125 n = 5 and p-values for Follow-Up

Sample no	k = 0 H=28.09	k= 3 H=6.80	k = 6 H= 3.44	p-value for each sample	
				Wilcoxon Test	AB Test
1	0.00	0	0	0.2234	0.1418
2	0.00	0	0	0.8416	0.7803
3	2.21	0	0	0.0412	0.8748
4	0.78	0	0	0.5014	0.7434
5	2.51	0	0	0.3930	0.0829
6	1.90	0	0	0.2448	0.8652
7	1.12	0	0	0.3385	0.5937
8	2.04	0	0	0.3700	0.1457
9	4.08	0	0	0.0564	0.5356
10	6.84	0	0	0.0372	0.5283
11	5.15	0	0	0.7666	0.6363
12	16.45	8.30	5.30	0.0027	0.0390
13	30.03	18.88	12.88	0.0016	0.0180
14	49.42	35.27	26.27	0.0005	0.0024
15	52.06	34.90	22.90	0.0389	0.5524

< Figure 1. Here>

The CL chart for k = 3 and 6 shows that the process stays in control for the first eleven test samples and goes OOC for the first time at sample number 12. The OOC signal persists in all the test samples from sample number 12 onwards till sample number 15. Note that, Mukherjee and Chakraborti (2012) and Chowdhury et al. (2013) also found first signal at the 12^{th} test sample. Following the signal from the chart at sample 12, it is of interest to see if the signal is due to a shift in location, scale or both. For this post signal follow-up diagnostic stage, we use step 8 of Section 4 and carry out a two-sided two-sample Wilcoxon rank-sum test first for location between the 125 observations from the reference sample (Phase I) and the 5 (Phase II) observations from the 12th test sample. This test yields a p-value p_1 = 0.0027. Next, we conduct a two-sided two-sample Ansari-Bradley test for scale using the same data and find the p-value as p_2 = 0.0390. Thus, while p_1 is much smaller than 1%, p_2 lies between 1% and 5%. Hence, we conclude that there is strong evidence of a shift in location with some evidence of a shift in scale at test sample number 12.

Next, we consider test sample number 13 which has been signaled to be OOC. If we carry out the same tests between the reference samples and test sample 13, we get p_1 = 0.001634 and p_2 = 0.01798, again indicating strong evidence of a shift in location with some evidence of a shift in scale. Note that, if k = 0 is used, the CL chart signals an OOC process from sample number 13 onwards without producing any signal at sample number 12 as in Mukherjee et al. (2013). This phenomenon of a delayed signal with $k = 0$ may be explained with our findings in Table 3 noting that the Piston ring data as in Montgomery (2005) is close to normal. Table 3 clearly shows that k = 3 is the best choice for detecting moderate to large shift in both location and scale.

Further, sample number fifteen shows OOC behavior irrespective of the choice of k and it is one such OOC signal which is not found in Mukherjee and Chakraborti (2012) and Chowdhury et al. (2013) in the context of joint monitoring with the same piston ring data. Note that none of the charts including the Shewhart in Montgomery (2005) has shown the fifteenth test sample to be OOC. However, Mukherjee et al. (2013) observed the same phenomenon at the 15th sample using CUSUM X-bar chart and Excedence CUSUM chart for detecting shift in the location parameter. In the context of joint monitoring, it is the CL chart which has been able to identify the fifteenth sample having come from an OOC process, although it is not known whether or not this is a genuine OOC signal or a false alarm. If we apply follow-up procedure for sample 15, we see while p_1 is marginally less than 5%, p_2 is significantly higher than 5%. Hence, we conclude that there is evidence of a shift in location and no evidence of a shift in scale at test sample number 15.

8. SUMMARY AND CONCLUSIONS

In this paper we consider a single phase II distribution-free CUSUM control chart based on the well-known Lepage (1972) statistic for joint monitoring of the location and scale parameters of a continuous distribution using a reference sample from a Phase I analysis. Our results show that the proposed chart has nice properties and is more effective than a competing distribution-free chart.

References:

Amin, R. W., Reynolds, M. R., Jr. and Bakir, S. T. (1995). Nonparametric quality control charts based on the sign statistic, *Communications in Statistics-Theory and Methods*, 24 (6), 1597-1623

Bakir, S.T. and Reynolds, M.R., Jr. (1979). A nonparametric procedure for process control based on within-group ranking. *Technometrics*, 21 (2), 175-183.

Brook, D. and Evans, D. A. (1972). An Approach to the Probability Distribution of Run Length, *Biometrika*, 59, 539- 549

Chakraborti, S. and Graham, M. A. (2007). Nonparametric control charts, Encyclopedia of Statistics in Quality and Reliability, 1, 415-429, John Wiley, New York.

Chakraborti, S., Human, S.W. and Graham, M. A. (2011). Nonparametric (distribution-free) quality control charts, In Handbook of Methods and Applications of Statistics: Engineering, Quality Control, and Physical Sciences. N. Balakrishnan, Ed., pp. 298-329, John Wiley & Sons, New York.

Chakraborti, S., Van der Laan, P. and Bakir, S. T. (2001). Nonparametric control charts: An overview and some results, *Journal of Quality Technology*, 33, 304-315.

Chang, T. C. and Gan, F. F. (1995). A cumulative sum control chart for monitoring process variance. *Journal of Quality Technology*, 27, 109-119.

Cheng, S. W. and Thaga, K. (2006). Single variables control charts: an overview, *Quality andReliability Engineering Internationa*l, 22, 811-820.

Chowdhury, S., Mukherjee, A. and Chakraborti, S. (2013). A New distribution-free control chart for joint monitoring of location and scale parameters of Continuous Distributions, *Quality and Reliability Engineering International*, DOI: 10.1002/qre.1488

Cucconi, O. (1968) Un Nuovo Test non Parametrico per ilConfrontotra Due GruppiCampionari, GiornaledegliEconomisti, XXVII, 225-248.

Gan, F. F. (1993). An optimal design of CUSUM control charts for binomial, *Journal of Applied Statistics*, 20 (4), 445-460.

Gan, F.F. (2007). Cumulative Sum (CUSUM) chart. *Encyclopedia of Statistics in Quality and Reliability*,1, 466-470, John Wiley, New York.

Gibbons, J. D. and Chakraborti, S. (2010). Nonparametric Statistical Inference, 5th ed., Taylor and Francis, Boca Raton, FL.

Goel, A. L. (2011). Cumulative sum control charts.In Handbook of Methods and Applications of Statistics: Engineering, Quality Control, and Physical Sciences. N. Balakrishnan, Ed., 120-129, John Wiley & Sons, New York.

Hawkins, D. M. (1987). Self-starting cusum charts for location and scale. *The Statistician*, 36, 299-315.

Hawkins, D.M. and Olwell, D.H. (1998). Cumulative sum charts and charting for quality improvement. Springer-Verlag, New York.

Jones, D.A. (1983). Statistical analysis of empirical models fitted by optimisation, *Biometrika*, 70 (1), 67–88

Jones, L. A., Champ, C. W. and Rigdon, S.E. (2004). The run length distribution of the CUSUM with estimated parameters, *Journal of Quality Technology*, 36 (1), 95-108.

Lepage, Y. (1971). A combination of Wilcoxon's and Ansari-Bradley's statistics, *Biometrika*, 58, 213-217.

Li, S.-Y., Tang, L. C. and Ng, S.-H. (2010). Nonparametric CUSUM and EWMA control charts for detecting mean shifts, *Journal of Quality Technology*, Vol 42 (2), 209-226,

Lucas, J.M. (1985). Cumulative sum (CUSUM) control schemes, *Communications in Statistics-Theory and Methods*, 14, 2689-2704.

McDonald, D. (1990). A CUSUM procedure based on sequential ranks, *Naval Research Logistics*, 37 (5), 627-646.

McCracken, A. K., and Chakraborti, S. (2013). Control Charts for Joint Monitoring of Mean and Variance: An Overview, Submitted to *Quality Technology and Quantitative Managament*, 10 (1), 17-36.

McCracken, A K., Chakraborti, S. and Mukherjee, A. (2013). Control Charts for Simultaneous Monitoring of Unknown Mean and Variance of Normally Distributed Processes, *Journal of Quality Technology*, 45 (4), 360-376.

Montgomery, D. C. (2005). Introduction to Statistical Quality Control, 5th ed., John Wiley, New York.

Mukherjee, A. and Chakraborti, S. (2012). A distribution-free control chart for joint monitoring of location and scale, *Quality and Reliability Engineering International*, 28, 335-352.

Mukherjee, A., Graham, M. A. and Chakraborti, S. (2013). Distribution-free exceedance CUSUM control charts for location, *Communications in Statistics - Simulation and Computation*, 42 (5), 1153-1187.

Page, E. S. (1954). Continuous inspection schemes, *Biometrika*, 41, 100-115.

Park C and Reynolds, M. R., Jr. (1987). Nonparametric procedures for monitoring a location parameter based on linear placement statistics, *Sequential Analysis*, 6 (4), 303-323

Reynolds, M. R., Jr., Amin, R.W. and Arnold, J. C. (1990). CUSUM charts with variable sampling intervals. *Technometrics*, 32 (4), 371-384

Ross, G. J., Tasoulis, D. K. and Adams, N. M. (2011). Nonparametric monitoring of data streams for changes in location and scale, *Technometrics*, 53 (4), 379-389.

Ross, G. J. and Adams, N. M. (2012). Two nonparametric control charts for detecting arbitrary distribution changes, *Journal of Quality Technology*, 44 (2), 1-15.

van der Vaart, A.W. (2000). Asymptotic Statistics, Cambridge University Press.

Woodall, W. H. (1983). The distribution of the run length of one-sided CUSUM procedures continuous random variables, *Technometrics*, 25, 295-301.

Yang, S. F. and Cheng, W. S. (2011). A new non-parametric CUSUM mean chart, *Quality and Reliability Engineering International*, 27 (7), 867-875.

Appendix

Conditional Distribution of S_L^2:

First note the following Lemma A.1. As a consequence of the Lemma A.1, we can easily obtain the required conditional distribution using preliminary concepts of distribution theory.

Lemma A.1: Given $\boldsymbol{X_m}$ **,** and as both *m*, *n* tends to ∞ such that $\frac{m}{m+n} \to \lambda$, asymptotic joint distribution of T_1 and T_2 is bi-variate normal with mean vector $\boldsymbol{\mu}(\boldsymbol{X_m})$ and variance − covariance matrix $\boldsymbol{\Sigma}(\boldsymbol{X_m})$, such that

$$\boldsymbol{\mu}(\boldsymbol{X_m}) = \left(\mu_1(\boldsymbol{X_m}), \mu_2(\boldsymbol{X_m})\right)' and \boldsymbol{\Sigma}(\boldsymbol{X_m}) = \begin{pmatrix} \sigma_1^2(\boldsymbol{X_m}) & \sigma_{12}(\boldsymbol{X_m}) \\ \sigma_{12}(\boldsymbol{X_m}) & \sigma_2^2(\boldsymbol{X_m}) \end{pmatrix},$$

Proof: Note that,

$$T_1 = \sum_{j=1}^{n}\sum_{i=1}^{m} I\left[Y_j > X_i\right] + \sum_{i=1}^{n}\sum_{j=1}^{n} I\left[Y_j > Y_i\right].$$

Therefore, given $\boldsymbol{X_m}$ **,** we have

$$T_1 = \sum_{j=1}^{n}\sum_{i=1}^{m} I\left[Y_j > x_i\right] + \frac{n(n+1)}{2} = n\sum_{i=1}^{m} \hat{\bar{G}}_n(x_i) + \frac{n(n+1)}{2}.$$

Note that

$$E(T_1|\boldsymbol{X_m} = \boldsymbol{x_m}) = \sum_{j=1}^{n}\sum_{i=1}^{m} \bar{G}(x_i) + \frac{n(n+1)}{2} = n\sum_{i=1}^{m} \bar{G}(x_i) + \frac{n(n+1)}{2},$$

and

$$V(T_1|\boldsymbol{X_m} = \boldsymbol{x_m}) = n^2 V\left[\sum_{i=1}^{m} \hat{\bar{G}}_n(x_i)\right] = n\left\{\sum_{i=1}^{m} \bar{G}(X_i)G(X_i) + \sum_{i=1}^{n}\sum_{j=1(\neq i)}^{m}\left[\bar{G}\left(X_i \wedge X_j\right) - \bar{G}(X_i)G\left(X_j\right)\right]\right\}.$$

Further note that, using Chernoff-Savage representation of Ansari-Bradley Statistics, we can express $(T_2|\boldsymbol{X_m} = \boldsymbol{x_m})$ by

$$\sum_{j=1}^{m}\left|\frac{m+n+1}{2} - j - n\hat{G}_n(x_{(j)})\right|,$$

where $x_{(j)}$ is the j-th order statistics of the observed sample $\boldsymbol{x_m}$.

Observe that, for any $i = 0,1,2,\ldots,n$,

$$Prob\left[n\hat{G}_n(x_{(j)}) = i\right] = \binom{n}{i}\left[G\left(X_{(j)}\right)\right]^i\left[\bar{G}\left(X_{(j)}\right)\right]^{n-i}.$$

Therefore, it is not difficult to obtain the desired expression for $E(T_2|\boldsymbol{X_m} = \boldsymbol{x_m})$ and $V(T_2|\boldsymbol{X_m} = \boldsymbol{x_m})$.

Further note that,

$$n\left\{\sum_{j=1}^{m}\hat{\hat{G}}_n(x_j)\right\}\times\left\{\sum_{j=1}^{m}\left|\frac{m+n+1}{2} - j - n\hat{G}_n(x_{(j)})\right|\right\}$$

$$= n\sum_{j=1}^{m}\hat{\hat{G}}_n(x_j)\left|\frac{m+n+1}{2} - j - n\hat{G}_n(x_{(j)})\right|$$

$$+\ n\sum_{j=1}^{m-1}\sum_{j'=j+1}^{m}\hat{\hat{G}}_n(x_j)\left|\frac{m+n+1}{2} - j' - n\hat{G}_n(x_{(j')})\right|$$

$$+ n\sum_{j=2}^{m}\sum_{j'=1}^{j-1}\hat{\hat{G}}_n(x_j)\left|\frac{m+n+1}{2} - j' - n\hat{G}_n(x_{(j')})\right|.$$

This provides us an expression for $E(T_1T_2 \mid \boldsymbol{X_m})$. Then, after a lengthy but straight forward computation, letting both m, n tends to ∞ such that $\frac{m}{m+n} \to \lambda$, for any arbitrary constant l_i, $(i = 1, 2\,)$, applying central limit theorem on

$$\frac{1}{\sqrt{m}}\left[l_1(T_1 - \mu_1(\boldsymbol{X_m})) + l_2(T_2 - \mu_2(\boldsymbol{X_m})\right]$$

given $\boldsymbol{X_m}$, and subsequently using Cramer-Wold device (see van der Vaart (2000, pp-16)), we can establish the Lemma A.

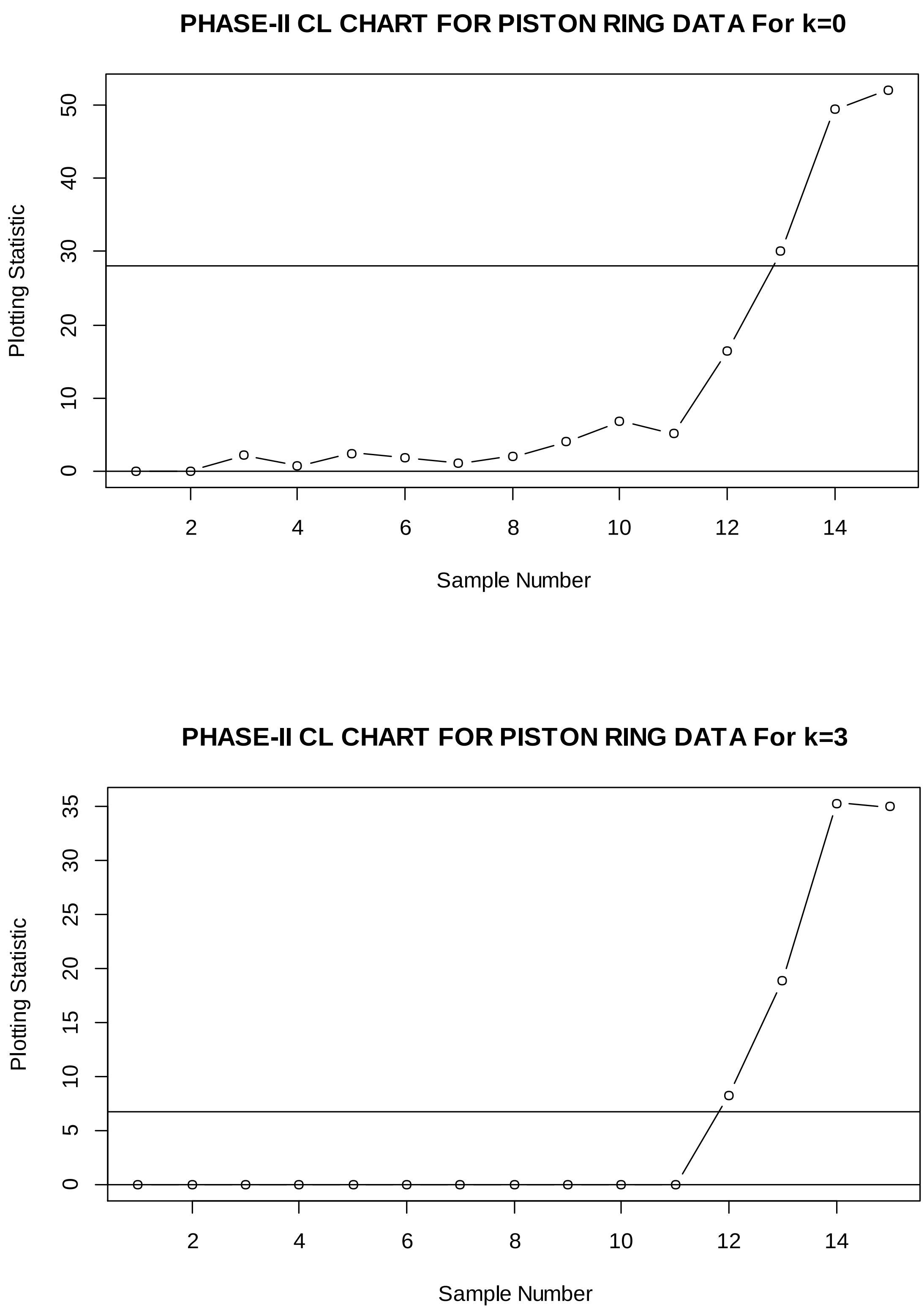
Table-1. Charting constant *H* for the CL chart, for various values of *m* and *n*, and for some standard (target) values of ARL_0

Chart Parameter	Reference Sample Size	Test Sample size	The Charting Constant (Upper Control Limit) : *H*		
k	*m*	*n*	Target $ARL_0 = 250$	Target $ARL_0 = 370$	Target $ARL_0 = 500$
0	30	5	13.548	15.529	17.183
		11	11.470	12.888	14.143
	50	5	16.705	18.919	21.188
		11	14.731	16.467	18.485
	100	5	20.227	23.730	26.551
		11	19.520	22.276	24.953
	150	5	22.320	26.062	29.432
		11	21.537	25.222	28.476
3	30	5	3.546	4.236	4.617
		11	3.607	4.089	4.457
	50	5	4.498	5.110	5.617
		11	4.326	4.908	5.366
	100	5	5.263	5.994	6.531
		11	5.197	5.882	6.435
	150	5	5.486	6.259	6.853
		11	5.507	6.277	6.859
6	30	5	0.543	1.150	1.568
		11	0.489	0.927	1.287
	50	5	1.277	1.961	2.379
		11	1.125	1.710	2.133
	100	5	2.016	2.689	3.279
		11	1.906	2.563	3.112
	150	5	2.214	2.989	3.596
		11	2.212	2.926	3.475

Table-2. IC performance characteristics of the CL chart for $ARL_0 = 500$

m	*n*	*H*	ARL_0	$SDRL_0$	5^{th} Percentile	1^{st} Quartile	Median	3^{rd} Quartile	95^{th} Percentile
Simulated values with *k*=0									
30	5	17.183	500.600	1049.699	13	40	108	365	2999
30	11	14.143	504.448	1066.059	8	27	87	372	3046
50	5	21.188	495.915	967.781	20	58	142	414	2523
50	11	18.485	503.024	997.102	13	42	122	419	2690
100	5	26.551	488.914	837.785	33	88	194	480	2052
100	11	24.953	508.989	889.857	26	75	179	494	2271
150	5	29.432	493.945	768.880	42	107	229	521	1888
150	11	28.476	504.454	810.384	35	95	215	528	2043
Simulated values with *k*=3									
30	5	4.6173	504.517	712.892	7	61	201	540	1952
30	11	4.4567	501.870	846.033	6	51	181	547	2200
50	5	5.617	500.868	787.318	10	72	217	566	1994
50	11	5.366	497.409	780.593	8	67	209	571	2019
100	5	6.531	506.972	695.658	15	100	267	620	1830
100	11	6.435	502.536	645.015	12	91	255	614	1876
150	5	6.853	501.232	612.544	17	108	281	636	1694
150	11	6.859	494.026	632.737	14	101	276	634	1733
Simulated values with *k*=6									
30	5	1.568	513.270	863.696	9	63	194	545	2204
30	11	1.287	497.279	817.785	8	58	192	553	2082
50	5	2.379	480.159	740.192	12	77	217	549	1873
50	11	2.133	501.533	759.076	11	74	224	590	1957
100	5	3.279	503.980	676.439	17	105	274	621	1801
100	11	3.111	506.052	678.493	16	97	269	634	1805
150	5	3.596	495.321	610.456	19	119	305	684	1754
150	11	3.475	503.862	641.108	17	107	285	644	1728

Figure-1.


PHASE-II CL CHART FOR PISTON RING DATA For k=0
Plotting Statistic
0
10
20
30
40
50
2
4
6
8
10
12
14
Sample Number
PHASE-II CL CHART FOR PISTON RING DATA For k=3
Plotting Statistic
0
5
10
15
20
25
30
35
2
4
6
8
10
12
14
Sample Number

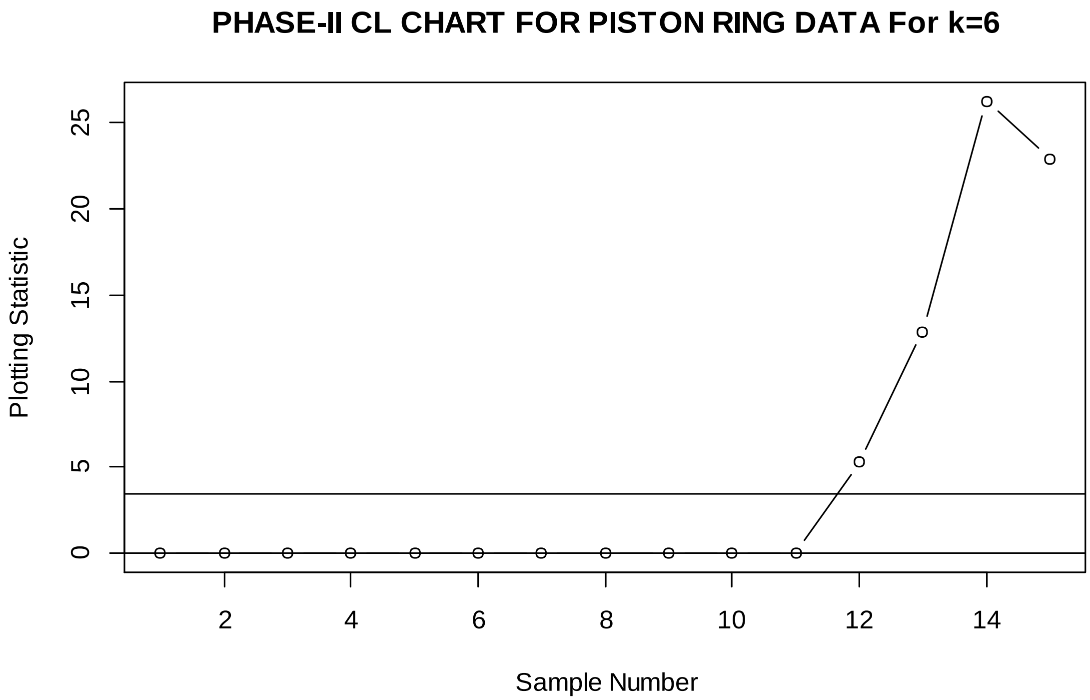

PHASE-II CL CHART FOR PISTON RING DATA For k=6
Plotting Statistic
0
5
10
15
20
25
2
4
6
8
10
12
14
Sample Number

Table-3. Performance comparisons between the Shewhart type and CUSUM type Lepage charts for the Normal (θ,δ) distribution with ARL_0 = 500.

θ	*m*=50, *n*=5				*m*=100, *n*=5			
	Shewhart Lepage Chart	Proposed CUSUM Lepage chart			Shewhart Lepage Chart	Proposed CUSUM Lepage chart		
		k=0	*k*=3	*k*=6		*k*=0	*k*=3	*k*=6
				δ =1.00				
0	499.6 (918.9) 13, 78, 215, 534, 1886	504.0 (978.7) 20, 58, 144, 422, 2523	503.9 (797.4) 10, 73, 216, 565, 2014	487.4 (753.4) 12, 78, 225, 555, 1890	513.0 (738.9) 18, 106, 276, 635, 1792	481.1 (824.2) 33, 87, 192, 472, 1966	471.3 (640.9) 15, 95, 252, 584, 1646	503.98 (676.4) 17, 105, 274, 621, 1801
0.25	292.7 (641.3) 7, 40, 116, 308, 1124	248.9 (645.3) 12, 30, 64, 165, 1058	291.2 (550.9) 5, 35, 107, 300, 1170	301.0 (554.9) 6, 39, 114, 317, 1186	257. 6 (410.3) 9, 47, 127, 303, 917	177.8 (407.7) 18, 40, 75, 157, 598	242.0 (397.4) 5, 40, 113, 280, 890	252.2 (399.8) 7, 45, 120, 296, 920
0.5	94.7 (253.9) 2, 12, 34, 91, 351	44.1 (141.6) 6, 13, 22, 40, 125	87.9 (211.1) 2, 10, 30, 83, 346	91.3 (209.1) 2, 12, 34, 89, 345	66.5 (98.6) 3, 13, 35, 80, 237	32.9 (37.8) 8, 16, 24, 38, 82	56.6 (92.1) 2, 10, 28, 65, 203	65.1 (100.2) 2, 13, 33, 77, 232
0.75	26.9 (61.1) 1, 5, 12, 28, 96	13.3 (12.5) 3, 7, 11, 16, 32	22.1 (57.20 1, 3, 9, 22, 82	24.7 (45.7) 1, 4, 11, 26, 92	20.3 (27.3) 1, 5, 12, 25, 68	13.2 (7.7) 4, 8, 12, 16, 27	14.9 (20.6) 1, 3, 8, 18, 51	18.9 (25.5) 1, 4, 10, 24, 64
1.0	9.3 (18.6) 1, 2, 5,11, 31	7.1 (4.4) 2, 4, 6, 9, 15	6.9 (10.9) 1, 2, 4, 8, 22	8.7 (14.1) 1, 2, 5, 10, 29	7.7 (8.8) 1, 2, 5, 10, 24	7.4 (3.6) 2, 5, 7, 9, 14	5.3 (5.6) 1, 2, 3, 7, 16	6.99 (8.2) 1, 2, 4, 9, 22
1.5	2.3 (2.2) 1, 1, 2, 3, 6	3.4 (1.4) 2, 2, 3, 4, 6	1.99 (1.5) 1, 1, 2, 2, 5	2.2 (1.98) 1, 1, 2, 3, 6	2.1 (1.7) 1, 1, 2, 3, 5	3.6 (1.4) 2, 2, 3, 4, 6	1.9 (1.1) 1, 1, 2, 2, 4	2.0 (1.5) 1, 1, 2, 2, 5
2	1.3 (0.6) 1, 1, 1, 1, 2	2.3 (0.7) 1, 2, 2, 3, 4	1.2 (0.5) 1, 1, 1, 1, 2	1.2 (0.6) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2	2.5 (0. 7) 2, 2, 2, 3, 4	1.2 (0.5) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2
3	1.0 (0.1) 1, 1, 1, 1, 1	1.6 (0.5) 1, 1, 2, 2, 2	1.0 (0.07) 1, 1, 1, 1, 1	1.0 (0.06) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1	2.0 (0.2) 2, 2, 2, 2, 2	1.0 (0.1) 1, 1, 1, 1, 1	1.0 (0.1) 1, 1, 1, 1, 1
				δ =1.25				
0	106.2 (197.8) 4, 21, 54 124, 369	64.7 (138.2) 9, 19, 34, 65, 200	95.1 (155.0) 2, 17, 47, 113, 340	103.7 (157.4) 3, 20, 53, 124, 366	102.9 (124.1) 5, 25, 62, 133, 337	54.8 (59.2) 12, 24, 39, 64, 144	86.1 (110.8) 2, 19, 50, 110, 291	99.8 (123.7) 4, 24, 60, 128, 330
0.25	73.6 (116.1) 3, 14, 37, 86, 261	46.4 (111.6) 7, 15, 26, 47, 131	70.3 (120.1) 2, 12, 33, 81, 256	74.5 (119.2) 2, 14, 37, 87, 270	70.6 (92.3) 3, 17, 41, 89, 232	39. 3 (36.7) 9, 19, 30, 47, 99	56.9 (75.7) 2, 12, 32, 71, 195	68.8 (89.5) 3, 16, 40, 87, 230
0.5	35.4 (55.5) 2, 7, 18, 42, 123	21.4 (30.3) 4, 10, 15, 25, 55	31.0 (54.5) 2, 6, 15, 35, 113	33.7 (51.4) 2, 7, 17, 40, 119	30.9 (38.8) 2, 8, 18, 40, 101	20.5 (14.0) 6, 12, 17, 25, 46	23.7 (32.8) 2, 5, 13, 29, 78	29.5 (37.3) 2, 7, 17, 38, 97
0.75	15.2 (22.5) 1, 4, 8, 18, 51	11.3 (8.5) 3, 6, 9, 14, 26	12.7 (19.5) 1, 3, 7, 15, 43	14.8 (21.5) 1, 3, 8, 18, 50	13.6 (15.6) 1, 4, 9, 18, 43	11.6 (6.5) 3, 7, 10, 15, 24	10.0 (11.8) 1, 3, 6, 13, 31	12.8 (14.99) 1, 3, 8, 17, 41
1.0	7.4 (9.3) 1, 2, 4, 9, 23	7.0 (4.2) 2, 4, 6, 9, 15	5.8 (7.1) 1, 2, 4, 7, 18	7.2 (9.2) 1, 2, 4, 9, 23	6.7 (7.0) 1, 2, 4, 9, 20	7.4 (3.7) 2, 5, 7, 10, 14	4.98 (4.9) 1, 2, 3, 6, 14	6.2 (6.7) 1, 2, 4, 8, 19
1.5	2.6 (2.4) 1, 1, 2, 3, 7	3.7 (1.7) 2, 2, 3, 5, 7	2.3 (1.7) 1, 1, 2, 3, 6	2.5 (2.2) 1, 1, 2, 3, 6	2.5 (2.1) 1, 1, 2, 3, 7	4.0 (1.7) 2, 3, 4, 5, 7	2.1 (1.4) 1, 1, 2, 3, 5	2.3 (1.9) 1, 1, 2, 3, 6
2	1.4 (0.9) 1, 1, 1, 2, 3	2.6 (0.9) 2, 2, 2, 3, 4	1.4 (0.7) 1, 1, 1, 2, 3	1.4 (0.8) 1, 1, 1, 2, 3	1.4 (0.8) 1, 1, 1, 2, 3	2.8 (0.9) 2, 2, 3, 3, 5	1.4 (0.6) 1, 1, 1, 2, 3	1.4 (0.7) 1, 1, 1, 2, 3
3	1.0 (0.2) 1, 1, 1, 1, 1	1.7 (0.5) 1, 1, 2, 2, 2	1.0 (0.2) 1, 1, 1, 1, 1	1.0 (0.15) 1, 1, 1, 1, 1	1.0 (0.2) 1, 1, 1, 1, 1	2.1 (0.3) 2, 2, 2, 2, 3	1.0 (0.2) 1, 1, 1 1, 1	1.0 (0.1) 1, 1, 1, 1, 1
				δ =1.50				
0	36.82 (46.98) 2, 9, 22, 47, 118	20.7 (17.7) 5, 10, 16, 25, 50	30.7 (42.99) 2, 6, 17, 38, 106	35.6 (46.3) 2, 8, 11, 45, 120	37.5 (42.2) 2, 10, 24, 50, 118	21.5 (13.3) 7, 13, 19, 27, 46	27.7 (33.2) 2, 7, 17, 36, 91	35.95 (41.7) 2, 9, 22, 47, 115
0.25	30.64 (38.81) 2, 7, 18, 39, 102	18.4 (15.8) 4, 9, 15, 23, 43	25.2 (35.4) 2, 5, 14, 31, 84	30.1 (39.3) 2, 7, 17, 38, 101	29.9 (34.2) 2, 8, 19, 39, 91	19.0 (11.4) 6,11, 17, 24, 40	22.2 (26.1) 2, 6, 14, 29, 71	27.97 (32.0) 2, 7, 18, 37, 89

0.5	19.0 (24.77) 1, 5, 11, 24, 64	13.4, (9.5) 3, 7, 11, 17, 30	15.4 (20.8) 1, 4, 9, 19, 52	18.6 (24.7) 1, 4, 11, 23, 62	17.8 (19.7) 1, 5, 12, 24, 55	13.9 (7.8) 4, 9, 12, 18, 28	13.2 (14.8) 1, 4, 8, 17, 42	16.99 (19.5) 1, 4, 11, 22, 54
0.75	10.78 (12.87) 1, 3, 7, 14, 34	9.2 (5.8) 2, 5, 8, 12, 20	8.5 (10.3) 1, 2, 5, 11, 27	10.4 (12.9) 1, 3, 6, 13, 33	10.2 (10.7) 1, 3, 7, 14, 30	9.9 (5.2) 3, 6, 9, 13, 19	7.5 (7.8) 1, 2, 5, 10, 23	9.6 (10.7) 1, 3, 6, 13, 30
1.0	6.5 (7.2) 1, 2, 4, 8, 19	6.7 (3.8) 2, 4, 6, 9, 14	5.0 (5.4) 1, 2, 3, 6, 15	5.98 (6.8) 1, 2, 4, 8, 18	6.1 (6.1) 1, 2, 4, 8, 18	7.2 (3.5) 2, 5, 7, 9, 14	4.6 (4.3) 1, 2, 3, 6, 13	5.6 (5.6) 1, 2, 4, 7, 17
1.5	2.8 (2.5) 1, 1, 2, 4, 8	4.0 (1.9) 2, 2, 4, 5, 8	2.4 (1.2) 1, 1, 2, 3, 6	2.7 (2.3) 1, 1, 2, 3, 7	2.7 (2.2) 1, 1, 2, 3, 7	4.3 (1.9) 2, 3, 4, 5, 8	2.3 (1.6) 1, 1, 2 3, 5	2.5 (2.0) 1, 1, 2, 3, 7
2	1.6 (1.1) 1, 1, 1, 2, 4	2.8 (1.1) 2, 2, 2, 3, 5	1.5 (0.86) 1, 1, 1, 2, 3	1.6 (0.99) 1, 1, 1, 2, 4	1.6 (1.0) 1, 1, 1, 2, 4	3.0 (1.1) 2, 2, 3, 4, 5	1.5 (0.8) 1, 1, 1, 2, 3	1.6 (0.9) 1, 1, 1, 2 3
3	1.1 (0.3) 1, 1, 1, 1, 2	1.8 (0.6) 1, 1, 2, 2, 3	1.1 (0.3) 1, 1, 1, 1, 2	1.1 (0.3) 1, 1, 1, 1, 2	1.1 (0.3) 1, 1, 1, 1, 2	2.2 (0.4) 2, 2, 2, 2, 3	1.1 (0.3) 1, 1, 1, 1, 2	1.1 (0.3) 1, 1, 1, 1, 2
				δ =1.75				
0	18.5 (20.7) 1, 5, 11, 24, 59	12.3 (7.7) 3, 7, 11, 16, 26	14.0 (17.2) 1, 4, 8, 18, 45	17.7 (21.1) 1, 4, 11, 23, 57	19.1 (20.3) 1, 5, 13, 26, 59	13.2 (6.8) 4, 9, 12, 17, 26	12.9 (14.1) 1, 4, 8, 17, 40	17.5 (19.0) 1, 5, 11, 24, 54
0.25	16.7 (20.4) 1, 4, 11, 22, 53	11.5 (7.1) 3, 7, 10, 14, 24	12.6 (14.9) 1, 3, 8, 16, 40	15.6 (18.3) 1, 4, 10, 20, 50	16.4 (17.2) 1, 5, 11, 22, 50	12.5 (6.4) 4, 8, 11, 16, 24,	11.5 (12.2) 1, 3, 8, 15, 35	15.5 (16.7) 1, 4, 10, 21, 48
0.5	12.1 (13.9) 1, 3, 8, 16, 37	9.6 (5.7) 3, 6, 9, 12, 20	9.3 (10.8) 1, 2, 6, 12, 29	11.6 (13.5) 1, 3, 7, 15, 37	12.1 (12.5) 1, 4, 8, 16, 37	10.5 (5.3) 3, 7, 10, 13, 20	8.6 (9.0) 1, 2, 6, 11, 26	11.2 (12.0) 1, 3, 7, 15, 35
0.75	8.3 (9.0) 1, 2, 5, 11, 25	7.8 (4.4) 2, 5, 7, 10, 16	6.4 (6.99) 1, 2, 4, 8, 20	7.9 (8.9) 1, 2, 5, 10, 25	8.4 (8.3) 1, 3, 6, 11, 24	8. 5 (4.2) 3, 6, 8, 11, 16	5.97 (5.8) 1, 2, 4, 8, 17	7.5 (7.6) 1, 2, 5, 10, 22
1.0	5.7 (5.8) 1, 2, 4, 7, 17	6.2 (3.3) 2, 4, 6, 8, 12	4.4 (4.3) 1, 2, 3, 6, 12	5.3 (5.6) 1, 2, 3, 7, 16	5.5 (5.2) 1, 2, 4, 7, 16	6.7 (3.2) 2, 4, 6, 8, 13	4.2 (3.7) 1, 2, 3, 5, 11	5.1 (4.9) 1, 2, 3, 7, 15
1.5	2.9 (2.6) 1, 1, 2, 4, 8	4.1 (1.9) 2, 2, 4, 5, 8	2.5 (1.9) 1, 1, 2, 3, 6	2.7 (2.4) 1, 1, 2, 3, 7	2.8 (2.4) 1, 1, 2, 4, 7	4.4 (1.9) 2, 3, 4, 6, 8	2.4 (1.7) 1, 1, 2, 3, 6	2.6 (2.1) 1, 1, 2, 3 7
2	1.8 (1.2) 1, 1, 1, 2, 4	3.0 (1.3) 2, 2, 3, 4, 5	1.7 (1.0) 1, 1, 1, 2, 4	1.7 (1.1) 1, 1, 1, 2, 4	1.8 (1.2) 1, 1, 1, 2, 4	3.2 (1. 3) 2, 2, 3, 4, 6	1.6 (0.9) 1, 1, 1, 2, 3	1.7 (1.1) 1, 1, 1, 2 4
3	1.1 (0.4) 1, 1, 1, 1, 2	2.0 (0.7) 1, 2, 2, 2, 3	1.1 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2	2.3 (0.6) 2, 2, 2, 3, 3	1.1 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2
				δ =2.00				
0	11.3 (12.3) 1, 3, 7, 15, 34	8.9 (4.9) 3, 6, 8, 11, 18,	8.4 (9.1) 1, 2, 5, 11, 26	10.8 (12.0) 1, 3, 7, 14, 34	11.5 (11.9) 1, 3, 8, 15, 35	9.8 (4.6) 3, 7, 9, 12, 18	7.8 (7.7) 1, 2, 5, 10, 23	10.6 (11.1) 1, 3, 7, 14, 32
0.25	10.3 (11.0) 1, 3, 7, 14, 32	8.6 (4.6) 2, 5, 8, 11, 17	7.6 (8.1) 1, 2, 5, 10, 23	9.9 (10.9) 1, 3, 6, 13, 31	10.8 (10.9) 1, 3, 7, 15, 32	9.5 (4.5) 3, 6, 9, 12, 18	7.3 (7.0) 1, 2, 5, 10, 21	9.8 (10.2) 1, 3, 6, 13, 20
0.5	8.5 (9.0) 1, 3, 6, 11, 25	7.7 (4.2) 2, 5, 7, 10, 16	6.4 (6.7) 1, 2, 4, 8, 19	8.2 (8.9) 1, 2, 5, 11, 25	8.6 (8.5) 1, 3, 6, 11, 25	8.5 (4.0) 3, 6, 8, 11, 16	6.1 (5.7) 1, 2, 4, 8, 17	7.98 (8.1) 1, 2, 5, 11, 24
0.75	6.5 (6.4) 1, 2, 4, 9, 19	6.7 (3.6) 2, 4, 6, 9, 13	5.1 (4.9) 1, 2, 3, 7, 14	6.3 (6.6) 1, 2, 4, 8, 19	6.6 (6.5) 1, 2, 5, 9, 19	7.3 (3.4) 2, 5, 7, 9, 14	4.8 (4.3) 1, 2, 3, 6, 13	6.0 (5.9) 1, 2, 4, 8, 18
1.0	4.9 (4.8) 1, 2, 3, 7, 14	5.7 (2.9) 2, 4, 5, 7, 11	3.9 (3.6) 1, 2, 3, 5, 11	4.7 (4.7) 1, 2, 3, 6, 14	4.8 (4.5) 1, 2, 3, 6, 14	6.2 (2.9) 2, 4, 6, 8, 11	3.7 (3.2) 1, 2, 3, 5, 10	4.5 (4.2) 1, 2, 3, 6, 13
1.5	2.9 (2.5) 1, 1, 2, 4, 8	4.1 (2.0) 2, 2, 4, 5, 8	2.5 (1.9) 1, 1, 2, 3, 6	2.7 (2.4) 1, 1, 2, 3, 7	2.9 (2.5) 1, 1, 2, 4, 8	4.5 (2.0) 2, 3, 4, 6, 8	2.4 (1.7) 1, 1, 2, 3, 6	2.7 (2.1) 1, 1, 2, 3, 7
2	1.9 (1.4) 1, 1, 1, 2, 5	3.1 (1.3) 2, 2, 3, 4, 6	1.7 (1.1) 1, 1, 1, 2 4	1.7 (1.1) 1, 1, 1, 2, 4, 16	1.9 (1.3) 1, 1, 1, 2, 4	3.4 (1.4) 2, 2, 3, 4, 6	1.7 (1.0) 1, 1, 1, 2 ,4	1.8 (1.2) 1, 1, 1, 2, 4
3	1.2 (0.5) 1, 1, 1, 1, 2	2.1 (0.8) 1, 2, 2, 2, 4	1.2 (0.4) 1, 1, 1, 1, 2	1.1 (0.4) 1, 1, 1, 1, 2	1.2 (0.5) 1, 1, 1, 1, 2	2.4 (0.7) 2, 2, 2, 3, 4	1.2 (0.4) 1, 1, 1, 1, 2	1.2 (0.4) 1, 1, 1, 1, 2

Table-4. Performance comparisons between the Shewhart type and CUSUM type Lepage charts for the Cauchy (θ,δ) distribution with ARL_0 = 500.

θ	m=50, n=5				m=100, n=5			
	Shewhart Lepage Chart	Proposed CUSUM Lepage chart			Shewhart Lepage Chart	Proposed CUSUM Lepage chart		
		k=0	k=3	k=6		k=0	k=3	k=6
δ =1.00								
0	468.9 (725.6) 13, 76, 212, 536, 1823	513.8 (993.4) 20, 58, 144, 427, 2605	487.7 (769.5) 10, 72, 211, 555, 1974	491.2 (759.0) 12, 79, 225, 565, 1890	502.1 (663.9) 18, 106, 277, 626, 1780	493.9 (859.8) 34, 88, 192, 474, 2038	478.8 (651.1) 14, 93, 255, 591, 1705	504.2 (676.7) 17, 104, 272, 625, 1784
0.25	468.3 (745.4) 11, 69, 203, 529,1869	426.5 (900.7) 17, 46, 113, 331, 2088	476.4 (767.6) 9, 66, 197, 531, 1921	472.4 (751.8) 10, 67, 198, 534, 1899	464.0 (642.2) 16, 90, 241, 570, 1681	389.8 (741.8) 27, 67, 145, 350, 1601	441.5 (630.1) 11, 80, 220, 538, 1611	451.5 (644.1) 14, 86, 229, 547, 1650
0.5	396.0 (690.8) 7,47, 149, 423, 1649	272.9 (690.5) 11, 29, 64, 183, 1198	408.8 (730.2) 5, 41, 145, 431, 1755	429.1 (756.2) 7, 48, 154, 453, 1835	379.4 (603.1) 10, 60, 172, 438,1445	207.1 (489.6) 17, 38, 73, 167, 765	346.8 (564.9) 6, 49, 148, 402, 1351	367.3 (584.3) 9, 58, 164, 420, 1405
0.75	343.6 (691.8) 4,29,98, 327, 1547	138.4 (456.9) 7, 17, 34, 82, 493	346.98 (726.4) 2, 21, 84, 312, 1611	340.1 (683.3) 4, 27, 97, 324, 1489	274.4 (507.0) 6, 35, 106, 287, 1093	86.2 (246.8) 10, 22, 38, 72, 258	229.9 (453.2) 3, 24, 78, 233, 956	265.0 (493.9) 5, 33, 101, 281, 1030
1.0	270.1 (641.5) 3,16, 58,211, 1257	60.7 (235.6) 5, 11, 20, 41, 180	258.6 (657.0) 2, 10, 42, 142, 1235	273.1 (643.5) 2, 15, 56, 214, 1295	173.3 (377.5) 3, 19, 57, 164, 719	35.8 (80.5) 7, 14, 22, 37, 96	140.9 (351.2) 2, 11, 37, 118, 598	173.9 (396.3) 2, 17, 53, 161, 721
1.5	149.2 (500.4) 1,5, 18, 73, 670	15.2 (27.6) 3, 6, 9, 16, 42	121.7 (473.6) 1, 3, 10, 41, 526	139.9 (470.4) 1, 5, 16, 67, 646	66.98 (214.7) 1, 6, 17, 50, 257	12.99 (13.3) 4, 7, 10, 15, 30	39.6 (162.3) 1, 3, 8, 24, 146	62.6 (198.4) 1, 5, 14, 46, 249
2	76.2 (347.2) 1,2, 7, 25, 283	7.8 (8.2) 2, 4, 6, 9, 19	52.5 (324.7) 1, 2, 4, 10, 122	72.5 (359.4) 1, 2, 6, 21, 243	26.5 (116.2) 1,2, 6, 17, 93	7.3 (4.3) 3, 5, 6, 9, 15	10.7 (62.7) 1, 2, 3, 7, 26	23.3 (127.4) 1, 2, 5, 13, 77
3	20.9 (179.4) 1, 1, 2, 5, 42	4.1 (2.5) 2, 3, 4, 5, 8	8.96 (121.2) 1, 1, 2, 3, 9	16.8 (146.4) 1, 1, 2, 4, 32	5.0 (22.3) 1, 1, 2, 4, 13	4.1 (1.7) 2, 3, 4, 5, 7	2.4 (7.1) 1, 1, 2, 2, 5	3.9 (18.9) 1, 1, 2, 3, 9
δ =1.25								
0	248.9 (440.7) 7, 40, 108, 271, 948	255.5 (657.1) 13, 32, 67, 173, 1080	250.5 (449.2) 5, 35, 102, 270, 940	245.7 (426.2) 6, 38, 108, 270, 940	238.9 (344.3) 9, 51, 128, 293, 810	192.7 (421.2) 19, 43, 82, 174, 664	216.8 (315.6) 6, 42, 114, 263, 773	239.2 (337.2) 8, 49, 129, 294, 836
0.25	232.9 (417.97) 6, 36, 101, 258, 879	231.3 (639.1) 11, 28, 60, 153, 989	240.2 (427.7) 4, 32, 94, 248, 911	235.6 (424.2) 5, 35, 99, 257, 894	221.8 (318.5) 8, 45, 117, 272, 775	161.7 (370.4) 16, 36, 68, 142, 566	195.6 (296.4) 4, 36, 99, 236, 704	216.6 (318.3) 7, 43, 116, 265, 741
0.5	205.2 (392.9) 5, 27, 80, 216, 819	164.4 (513.5) 8, 20, 41, 99, 595	214.5 (439.7) 2, 22, 73, 215, 877	211.9 (412.2) 4, 26, 79, 219, 846	185.4 (290.1) 6, 34, 91, 217, 670	107.1 (289.97) 12, 26, 46, 90, 330	160.2 (264.2) 3, 25, 73, 184, 600	179.5 (282.4) 5, 32, 87, 213, 643
0.75	178.9 (397.5) 3, 18, 57, 167, 738	95.6 (336.6) 6, 14, 26, 58, 320	175.3 (422.6) 2, 14, 46, 151, 745	179.6 (393.2) 2, 17, 56, 170, 739	135.8 (234.9) 4, 22, 60, 152, 516	53.2 (119.3) 9, 18, 29, 52, 151	114.4 (217.7) 2, 25, 44, 119, 454	135.9 (254.6) 3, 20, 57, 148, 525
1.0	148.1 (383.4) 2,11,37, 117, 620	53.1 (233.5) 5, 10, 18, 34, 142	136.9 (391.6) 2, 8, 27, 98, 593	144.96 (371.4) 2, 11, 36, 118, 613	97.4 (196.1) 3, 14, 38, 99, 385	30.2 (77.7) 6, 12, 19, 31, 76	71.9 (169.1) 2, 8, 24, 68, 288	93.0 (189.4) 2, 12, 35, 94, 368
1.5	81.98 (271.8) 1, 5, 15, 50, 353	16.4 (58.2) 3, 6, 9, 15, 41	72.6 (310.40 1, 3, 9, 32, 277	83.7 (312.8) 1, 4, 13, 47, 332	42.9 (122.1) 1, 5, 14, 37, 159	12.5 (10.5) 4, 7, 10, 15, 28	25.7 (86.5) 1, 3, 8, 20, 93	40.4 (123.2) 1, 4, 12, 34, 151
2	46.6 (232.2) 1, 2, 6, 20, 150	8.1 (9.9) 2, 4, 6, 9, 19	31.2 (195.8) 1, 2, 4, 10, 86	45.8 (233.1) 1, 2, 6, 18, 162	18.7 (67.2) 1, 3, 6, 15, 65	7.6 (4.2) 3, 5, 7, 9, 15	8.8 (34.3) 1, 2, 4, 8, 26	16.3 (55.8) 1, 2, 5, 13, 58
3	14.1 (106.2) 1, 1, 2, 5, 33	4.3 (2.6) 2, 3, 4, 5, 8	6.95 (74.8) 1, 1, 2, 3, 10	14.1 (130.1) 1, 1, 2, 4, 26	4.3 (12.4) 1, 1, 2, 4, 13	4.3 (1.8) 2, 3, 4, 5, 8	2.5 (4.4) 1, 1, 2, 3, 6	3.9 (36.8) 1, 1, 2, 3, 10
δ =1.50								
0	138.2 (243.0) 4, 24, 65, 155, 493	117.0 (393.6) 9, 20, 36, 75, 372	134.9 (273.6) 2, 19, 55, 144, 502	138.9 (251.2) 4, 23, 63, 153, 514	131.3 (175.6) 6, 29, 75, 166, 444	72.9 (149.2) 13, 25, 42, 73, 212	108.9 (156.7) 3, 22, 58, 135, 375	127.2 (175.1) 4, 27, 71, 159, 438
0.25	133.5 (248.7) 4, 22, 59, 145, 497	103.7 (341.7) 8, 18, 33, 71, 341	128.8 (250.6) 2, 17, 51, 135, 493	133.1 (245.9) 3, 21, 58, 146, 495	122.6 (167.4) 5, 27, 68, 154, 417	66.7 (156.3) 11, 23, 38, 66, 185	103.3 (153.6) 2, 19, 52, 125, 367	120.2 (172.5) 4, 25, 66, 147, 417

0.5	122.7 (245.4) 3,18, 50,127, 463	79.5 (275.4) 7, 15, 27, 54, 251	115.7 (253.4) 2, 14, 42, 116, 447	120.6 (237.3) 3, 17, 49, 127, 464	105.9 (156.9) 4, 21, 56, 128, 372	48.6 (91.7) 10, 19, 30, 51, 133	86.6 (142.9) 2, 15, 41, 102, 315	101.9 (155.2) 3, 20, 53, 122, 357
0.75	103.8 (224.6) 2, 13, 37, 104, 406	58.0 (240.9) 5, 12, 20, 39, 158	96.8 (237.6) 2, 9, 29, 87, 396	103.2 (230.6) 2, 12, 37, 103, 403	84.4 (134.4) 3, 16,41, 99, 304	33.9 (58.5) 7, 14, 22, 37, 87	64.3 (110.3) 2, 10, 29, 71, 246	78.9 (131.4) 2, 13, 37, 91, 291
1.0	84.1 (218.2) 2, 9, 26, 75, 332	36.1 (161.5) 4, 9, 15, 27, 93	74.6 (209.7) 2, 6, 19, 59, 307	84.8 (209.5) 2, 9, 26, 78, 341	61.3 (116.2) 2, 10, 28, 66, 225	22.5 (48.8) 6, 11, 16, 25, 54	44.8 (93.6) 2, 6, 18, 46, 174	59.96 (116.5) 2, 9, 25, 64, 221
1.5	51.3 (155.9) 1, 4, 12, 37, 206	14.8 (56.1) 3, 6, 9, 14, 35	43.4 (171.1) 1, 3, 8, 24, 165	52.9 (179.8) 1, 4, 11, 37, 214	30.0 (65.2) 1, 5, 12, 30, 112	11.9 (9.2) 4, 7, 10, 14, 26	18.4 (51.6) 1, 3, 7, 17, 66	28.2 (77.0) 1, 4, 11, 28, 104
2	31.3 (142.8) 1, 2, 6, 18, 112	8.1 (9.6) 2, 4, 6, 9, 19	20.2 (124.5) 1, 2, 4, 10, 61	30.8 (144.5) 1, 2, 5, 16, 107	14.5 (38.1) 1, 3, 6, 14, 52	7.7 (4.4) 3, 5, 7, 9, 15	7.6 (20.1) 1, 2, 4, 8, 23	12.9 (36.2) 1, 2, 5, 12, 45
3	12.1 (98.8) 1, 1, 2, 5, 29	4.4 (2.7) 2, 3, 4, 5, 9	6.5 (86.1) 1, 1, 2, 3, 11	10.3 (79.99) 1, 1, 2, 5, 24	4.4 (10.3) 1,1,2, 5, 13	4.5 (1.9) 2, 3, 4, 5, 8	2.6 (3.7) 1, 1, 2, 3, 6	3.7 (8.8) 1, 1, 2, 4, 10
				δ =1.75				
0	85.6 (151.3) 3, 15, 41, 98, 304	52.7 (182.7) 6, 14, 23, 42, 140	76.97 (151.8) 2, 12, 33, 84, 290	86.2 (155.2) 2, 15, 40, 97, 313	81.1 (104.2) 4,19, 47, 104, 272	32.2 (58.4) 9, 17, 27, 42, 94	63.3 (96.8) 2, 13, 34, 79, 220	79.8 (109.4) 3, 18, 45, 99, 271
0.25	83.6 (149.9) 3, 15, 39, 94, 306	50.97 (190.6) 6, 13, 22, 40, 139	75.7 (154.6) 2, 11, 31, 80, 287	82.8 (148.9) 2, 14, 38, 92, 299	78.6 (105.9) 4, 18, 45, 98, 262	34.4 (43.7) 9, 16, 25, 39, 84	60.3 (89.7) 2, 12, 31, 73, 212	75.9 (109.1) 2, 16, 42, 94, 258
0.5	79.6 (155.9) 3, 13, 34, 84, 299	43.5 (181.5) 6, 12, 19, 35, 110	69.0 (148.5) 2, 9, 26, 70, 268	76.1 (143.5) 2, 12, 33, 83, 281	68.9 (95.4) 3, 15, 38, 85, 233	29.4 (36.8) 7, 15, 22, 34, 71	52.2 (82.1) 2, 10, 26, 61, 190	64.4 (88.9) 2, 13, 35, 80, 223
0.75	68.4 (140.5) 2, 10, 27, 71, 257	34.5 (150.2) 5, 10, 16, 28, 84	58.8 (143.3) 2, 7, 20, 56, 230	67.9 (142.7) 2, 9, 27, 69, 261	56.2 (81.9) 2, 12, 30, 68, 197	23.3 (27.8) 6, 12, 18, 27, 55	40.8 (71.7) 2, 7, 19, 46, 149	54.8 (85.7) 2, 10, 27, 64, 197
1.0	57.6 (143.1) 2, 7,20, 55, 216	24.7 (101.7) 4, 8, 13, 22, 61	46.6 (117.9) 2, 5, 14, 41, 188	57.6 (136.3) 2, 7, 20, 55, 224	43.8 (70.4) 2, 9, 22, 51, 154	18.1 (21.9) 5, 10, 15, 22, 40	28.8 (52.7) 2, 5, 13, 31, 104	41.4 (71.5) 2, 7, 20, 47, 153
1.5	38.2 (112.4) 1, 4, 11, 31, 149	13.5 (52.4) 3, 6, 9, 14, 30	29.0 (102.5) 1, 3, 7, 20, 108	35.1 (101.2) 1, 4, 10, 28, 138	23.5 (41.7) 1, 5, 11, 26, 83	11.1 (7.3) 4, 7, 10, 13, 23	14.2 (30.4) 1, 3, 6, 15, 49	22.1 (43.6) 1, 4, 10, 23, 81
2	22.7 (90.4) 1, 2, 6, 16, 81	8.1 (11.3) 2, 4, 6, 9, 18	15.5 (75.5) 1, 2, 4, 9, 49	21.9 (89.1) 1, 2, 5, 15, 80	12.9 (26.1) 1, 3, 6, 13, 45	7.6 (4.0) 3, 5, 7, 9, 15	7.1 (15.7) 1, 2, 4, 7, 21	11.4 (24.3) 1, 2, 5, 12, 39
3	8.9 (50.4) 1, 1, 2, 6, 24	4.6 (3.3) 2, 3, 4, 6, 9	5.5 (53.7) 1, 1, 2, 4, 11	8.8 (71.97) 1, 1, 2, 5, 21	4.3 (7.9) 1, 1, 2, 5, 13	4.6 (1.99) 2, 3, 4, 6, 8	2.8 (3.3) 1, 1, 2, 3, 7	3.7 (6.5) 1, 1, 2, 4, 11
				δ =2.00				
0	57.2 (94.8) 2,11, 29, 66, 204	30.7 (115.4) 5, 11, 17, 29, 75	49.3 (92.8) 2, 8, 22, 53, 183	57.8 (100.1) 2, 10, 28, 67, 206	56.6 (70.8) 3, 14, 33, 72, 188	24.5 (25.6) 7, 13, 20, 29, 56	38.96 (53.7) 2, 8, 21, 48, 134	52.8 (68.0) 2, 12, 31, 68, 177
0.25	56.6 (97.7) 2, 11, 28, 65, 198	28.2 (91.7) 5, 10, 17, 28, 70	47.95 (92.0) 2, 7, 20, 50, 178	56.7 (105.1) 2, 10, 27, 64, 199	52.98 (68.1) 3, 12, 31, 68, 173	23.4 (24.2) 7, 13, 19, 28, 52	37.7 (52.7) 2, 8, 21, 46, 131	50.4 (72.3) 2, 11, 29, 63, 172
0.5	53.6 (98.1) 2, 10, 25, 60, 193	25.4 (65.6) 5, 10, 15, 25, 65	44.5 (98.7) 2, 6, 18, 45, 167	52.4 (96.4) 2, 9, 24, 58, 196	48.5 (63.3) 2, 11, 27, 61, 165	21.1 (18.4) 6, 12, 17, 25, 47	33.2 (49.7) 2, 7, 17, 40, 116	45.6 (61.9) 2, 10, 25, 57, 156
0.75	47.6 (92.3) 2, 8, 21, 51, 174	21.3 (62.95) 4, 9, 13, 22, 53	38.2 (84.7) 2, 5, 15, 38, 140	45.6 (82.2) 2, 7, 20, 49, 169	40.9 (57.2) 2, 9, 22, 50, 141	18.3 (16.2) 6, 10, 15, 22, 41	27.6 (46.3) 2, 6, 14, 32, 97	38.2 (56.4) 2, 8, 21, 46, 132
1.0	40.3 (86.7) 2, 6, 16, 41, 149	17.99 (62.98) 4, 7, 11, 18, 43	32.4 (80.7) 2, 4, 11, 30, 128	40.99 (94.4) 1, 6, 16, 40, 152	33.3 (51.2) 2, 7, 17, 40, 115	15.0 (10.4) 5, 9, 13, 18, 33	20.98 (34.8) 2, 4, 10, 24, 74	31.1 (50.8) 2, 6, 16, 37, 110
1.5	28.6 (73.9) 1, 4, 10, 26, 109	11.2 (22.9) 3, 6, 8, 12, 26	20.4 (56.6) 1, 2, 6, 16, 79	27.4 (75.4) 1, 3, 9, 24, 104	19.89 (33.2) 1, 4, 10, 23, 69	9.2 (6.2) 4, 6, 9, 13, 21	11.6 (19.5) 1, 3, 6, 13, 39	17.9 (33.8) 1, 3, 9, 20, 62
2	18.2 (70.0) 1, 2, 6, 14, 66	7.7 (9.6) 2, 4, 6, 9, 17	12.1 (49.5) 1, 2, 4, 9, 38	16.8 (61.1) 1, 2, 5, 13, 61	11.5 (19.8) 1, 3, 6, 13, 39	7.5 (3.9) 3, 5, 7, 9, 15	6.4 (9.9) 1, 2, 4, 7, 20	10.3 (22.7) 1, 2, 5, 11, 35
3	7.6 (28.1) 1,1, 3, 6, 24	4.7 (2.8) 2, 3, 4, 6, 9	5.3 (43.8) 1, 1, 2, 4, 11	6.9 (34.0) 1, 1, 2, 5, 20	4.6 (7.7) 1, 1, 3, 5, 14	4.8 (2.1) 2, 3, 4, 6, 9	2.8 (2.9) 1, 1, 2, 3, 7	3.8 (5.9) 1, 1, 2, 4, 11