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Stochastic Comparisons of Parallel Systems of Heterogeneous Generalized Exponential Components

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Abstract

Let $X_1, X_2, \ldots, X_n$ (resp. $Y_1, Y_2, \ldots, Y_n$) be independent random variables such that X_i (resp. Y_i) follows generalized exponential distribution with shape parameter θ_i and scale parameter λ_i (resp. δ_i), $i = 1, 2, \ldots, n$. Here it is shown that if $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ majorizes $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$ then $X_{n:n}$ will be greater than $Y_{n:n}$ in reversed hazard rate ordering. That no relation exists between $X_{n:n}$ and $Y_{n:n}$, under same condition, in terms of likelihood ratio ordering has also been shown. It is also shown that, if Y_i follows generalized exponential distribution with parameters $\left(\overline{\lambda}, \theta_i\right)$, where $\overline{\lambda}$ is the mean of all λ_i's, $i = 1 \ldots n$, then $X_{n:n}$ is greater than $Y_{n:n}$ in likelihood ratio ordering. In this context, an error in Marshall, Olkin and Arnold [Inequalities: Theory of Majorization and Its applications (2011)] has been corrected, and some new results on majorization have been developed.

Keywords and Phrases: Hazard rate function, majorization, reversed hazard rate function, Schur-convex and Schur-concave functions, stochastic orders.
AMS 2010 Subject Classifications: 60E15, 60K10

1 Introduction

Let $X_1, X_2, \ldots, X_n$ be the lifetimes of the n components forming the system under consideration. We write $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ to represent order statistics from $X_1, X_2, \ldots, X_n$. Sometimes we write $X_{(i)}$ to represent $X_{i:n}$ if there is no ambiguity. Order statistics play an important role in statistics, reliability theory, applied probability and many other related fields. There is a one-to-one correspondence between order statistics and a k-out-of-n system. It is a system which functions if and only if at least k of the n components function. The time to failure of a k-out-of-n system with component lifetimes X_1, X_2, …, X_n corresponds to the $(n-k+1)^{th}$ order statistic $X_{n-k+1:n}$. Series system and parallel system are particular cases of a k-out-of-n system. A series system is an n-out-of-n system and a parallel system is a 1-out-of-n system. Order statistics have been extensively studied in the case when the observations are independent and identically distributed (i.i.d.). Due to the complicated expressions of the distributions in the non-i.i.d. case, only limited results are found in the literature. One may refer to David and Nagaraja (2003), Balakrishnan and Rao (1998a, 1998b) for results on the independent and non-identically (i.ni.d.) distributed random variables.

Due to nice mathematical form and the memoryless property, the exponential distribution has widely been applied in statistics, reliability, operations research, life testing and many other areas. One may refer to Barlow and Proschan (1975), Balakrishnan and Basu (1995), Dykstra et al. (1997), and Khaledi and Kochar (2000) for the study on exponential distribution. Zhao et al. (2009) and Zhao and Balakrishnan (2012) studied stochastic comparison of the second order statistics from independent heterogeneous exponential random variables as well as multiple-outlier heterogeneous exponential random variables. Recently, Torrado and Kochar (2014) also studied the same for Weibull and multiple-outlier Weibull models. Gupta and Kundu (1999) defined generalized exponential distribution (GE) and proved some interesting results of this distribution. A random variable X is said to have generalized exponential (also known as exponentiated exponential) distribution if the distribution function of X is given by

$$F(x; \lambda, \theta) = \left(1 - e^{-\lambda x}\right)^{\theta}, \quad x > 0, \lambda, \theta > 0, \tag{1.1}$$

where θ is the shape parameter and λ is the scale parameter. Clearly, this distribution is a generalization of exponential distribution in the sense that one can obtain exponential distribution from this distribution by taking $\theta = 1$. They also pointed out that unlike exponential distribution this distribution has increasing (decreasing) failure rate for $\theta > (<)1$ for any fixed λ. Therefore, if it is known that the data are from a regular maintenance environment, it may make more sense to fit generalized exponential distribution than exponential distribution. In this paper our main aim is to compare two parallel systems in terms of reversed hazard rate order and likelihood ratio order when the components are from two heterogeneous generalized exponential distributions. In the process of developing the results, we correct a mistake in Marshall et al. (2011). We also discuss some new findings in the theory of majorization. The organization of the paper is as follows.

In Section 2, we give different notations and definitions used in this paper. Results related to reversed hazard rate (rh) order and likelihood ratio (lr) order between two order statistics $X_{n:n}$ and $Y_{n:n}$, where the components are from two i.ni.d. generalized exponential distributions are given in Section 3. We have shown that there exists lr ordering between $X_{2:2}$ and $Y_{2:2}$ and one counterexample is also provided here to show that no comparison between $X_{3:3}$ and $Y_{3:3}$ can be made in terms of lr ordering. It is also shown that, under some relaxation of the conditions on the parameters, there may exists lr order between $X_{n:n}$ and $Y_{n:n}$. lr ordering between $X_{n:n}$ and $Y_{n:n}$ is also established for the case of multiple-outlier GE model. Throughout the paper the word increasing (resp. decreasing) and nondecreasing (resp. nonincreasing) are used interchangeably, and $\Re$ denotes the set of real numbers $\{x : -\infty < x < \infty\}$.

2 Notations, Definitions and Preliminaries

Let X and Y be two nonnegative random variables having distribution functions F and G, probability density functions f and g, hazard rate functions r_X and r_Y, and reversed hazard rate functions $\overline{r}_X$ and $\overline{r}_Y$ respectively. Further, write $\overline{F} \equiv 1 - F$ and $\overline{G} \equiv 1 - G$ as the corresponding survival functions. The following well known definitions may be obtained in Shaked and Shanthikumar (2007).

Definition 2.1 *X is said to be smaller than Y in*

(i) likelihood ratio ordering (written as $X \leq_{lr} Y$) if $\frac{g(x)}{f(x)}$ is increasing in $x \geq 0$;

(ii) hazard rate ordering (written as $X \leq_{hr} Y$) if $r_X(x) \geq r_Y(x)$, for all $x \geq 0$;

(iii) reversed hazard rate ordering (written as $X \leq_{rh} Y$) if $\overline{r}_X(x) \leq \overline{r}_Y(x)$, for all $x \geq 0$.

It can be checked that (iii) holds if and only if

$$\frac{G(x)}{F(x)} \text{ is increasing in } x \geq 0. \tag{2.1}$$

It is well known that the notion of majorization is quite useful in establishing various inequalities. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ be the increasing arrangements and $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ be the decreasing arrangements of the components of the vector $\mathbf{x}=(x_1,\ x_2, \ldots, x_n)$. The following definitions may be obtained in Marshall et al. (2011).

Definition 2.2 *Let I^n denote an n-dimensional Euclidean space where $I \subseteq \Re$.*

(i) A point $\mathbf{x}=(x_1,\ x_2, \ldots, x_n) \in I^n$ is said to majorize another point $\mathbf{y}=(y_1,\ y_2, \ldots, y_n) \in I^n$ (written as $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} y_{[i]},\ j = 1,\ 2,\ \ldots, n-1, \ \text{and} \ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]} \tag{2.2}$$

or equivalently

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)},\ j = 1,\ 2,\ \ldots, n-1, \ \ and \ \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)} \tag{2.3}$$

(ii) A function $\psi : I^n \rightarrow \Re$ *is said to be Schur-convex (Schur-concave) on* I^n *if* $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$ *implies*

$$\psi(\mathbf{x}) \geq (\leq) \psi(\mathbf{y}) \ \ for\ all\ \mathbf{x},\ \mathbf{y} \in I^n. \tag{2.4}$$

Notation 2.1 *Let us define the following notations. The first two are borrowed from Marshall et al. (2011).*

(i) $\mathcal{D} = \{(x_1, x_2, \ldots, x_n) : x_1 \geq x_2 \geq \ldots \geq x_n\}$.

(ii) $\mathcal{D}_+ = \{(x_1, x_2, \ldots, x_n) : x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}$.

(iii) $\mathcal{D}^+ = \{(x_1, x_2, \ldots, x_n) : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n\}$

3 Main Results

Let X_i (resp. Y_i), $i = 1, 2, \ldots, n$ be n independent random variables following generalized exponential distribution having distribution function $F(\cdot; \lambda_i, \theta_i)$ (resp. $F(\cdot; \delta_i, \theta_i)$), for all $i = 1, 2, \ldots, n$, where $F(\cdot; \lambda, \theta)$ is as given in (1.1). Let $F_{n:n}(\cdot; \boldsymbol{\lambda}, \boldsymbol{\theta})$, $G_{n:n}(\cdot; \boldsymbol{\delta}, \boldsymbol{\theta})$ be the distribution functions of $X_{n:n}$ and $Y_{n:n}$ respectively, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{D}_+$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathcal{D}_+$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathcal{D}^+$. Then, clearly

$$F_{n:n}(x; \boldsymbol{\lambda}, \boldsymbol{\theta}) = \prod_{i=1}^{n} F(x; \lambda_i, \theta_i) = \prod_{i=1}^{n} \left(1 - e^{-\lambda_i x}\right)^{\theta_i},$$

and

$$G_{n:n}(x; \boldsymbol{\delta}, \boldsymbol{\theta}) = \prod_{i=1}^{n} F(x; \delta_i, \theta_i) = \prod_{i=1}^{n} \left(1 - e^{-\delta_i x}\right)^{\theta_i}.$$

The following proposition may be obtained in Marshall et al. (2011, p. 84).

Proposition 3.1 *Let* $I \subset \Re$ *be an open interval and let* $\phi : I^n \rightarrow \Re$ *be continuously differentiable. Then, for* ϕ *to be Schur-concave on* I^n*, the conditions*

$$\phi \ is\ symmetric\ on\ I^n \tag{3.1}$$

and

$$\phi_{(i)}(\mathbf{z}) \ is\ increasing\ in\ i = 1, 2, \ldots, n\ for\ all\ \mathbf{z} = (z_1, z_2, \ldots, z_n) \in D \cap I^n$$

are necessary as well as sufficient, where $\phi_{(i)}(\mathbf{z}) = \frac{\partial}{\partial z_i}\phi(\mathbf{z})$.

Alternatively, a function ϕ *in* I^n *is Schur-concave if and only if (3.1) holds and*

$$(z_i - z_j)(\phi_{(i)}(\mathbf{z}) - \phi_{(j)}(\mathbf{z})) \leq 0, \ for\ all\ \mathbf{z} \in I^n \cap D,$$

for $i \neq j$. □

On using the above proposition, we have the following lemma.

Lemma 3.1 *Let $\phi(\mathbf{x}) = \sum_{i=1}^{n} g_i(x_i)$, $x \in D_+$, where $g_i : \Re \to \Re$ is differentiable, for all $i = 1, 2, \ldots, n$. Then ϕ is Schur-concave if and only if $g_i'(a) \le g_{i+1}'(b)$ whenever $a \ge b$, for all $i = 1, 2, \ldots, n-1$, where $g'(a) = \frac{dg(x)}{dx}\big|_{x=a}$.* □

Now, we are in a position to prove the following theorem.

Theorem 3.1 *Let, $\phi(\mathbf{x}) = \sum_{i=1}^{n} u_i g(x_i)$ where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and let $I \subset \Re$ be an interval. Consider a function $g : I \to \Re$.*

(a) If $u \in \mathcal{D}_+$ and

- *(i) $g(\cdot)$ is increasing and convex then ϕ is Schur-convex;*
- *(ii) $g(\cdot)$ is decreasing and concave then ϕ is Schur-concave.*

(b) If $u \in \mathcal{D}^+$ and

- *(i) $g(\cdot)$ is increasing and concave then ϕ is Schur-concave;*
- *(ii) $g(\cdot)$ is decreasing and convex then ϕ is Schur-convex.*

Proof:

(a) Let $g_i(x_i) = u_i g(x_i)$. If $g(\cdot)$ is increasing and convex, then for all $a \ge b$, $g^{'}(a) \ge g^{'}(b) \ge 0$. Again, if $u \in \mathcal{D}_+$ then $u_i \ge u_{i+1} \ge 0$ and consequently, $u_i g^{'}(a) \ge u_{i+1} g^{'}(b)$. Hence, by Proposition H.2 of Marshall et al. (2011), (i) is proved. To prove (ii), as $g(\cdot)$ is decreasing, let $-g^{'}(x) = G(x) > 0$ for all x. Again, as $g(\cdot)$ is concave then for all $a \ge b$, $g^{'}(a) \le g^{'}(b)$ implying that $G(a) \ge G(b)$. So, if $u \in \mathcal{D}_+$ then $u_i G(a) \ge u_{i+1} G(b)$ and consequently $u_i g^{'}(a) \le u_{i+1} g^{'}(b)$. Hence, by Lemma 3.1, the result is proved.

(b) Note that $g(\cdot)$ is increasing and concave implies that for all $a \ge b$, $g^{'}(a) > 0, g^{'}(b) > 0$ and $g^{'}(a) \le g^{'}(b)$. So, if $u \in \mathcal{D}^+$ and consequently $u_i \le u_{i+1}$, then $u_i g^{'}(a) \le u_{i+1} g^{'}(b)$. Hence, by Lemma 3.1, ($i$) is proved. If $g(\cdot)$ is decreasing and convex, then for all $a \ge b$, $g^{'}(a) \ge g^{'}(b)$ and consequently $G(a) \le G(b)$. So, $u \in \mathcal{D}^+$ implies that $u_i G(a) \le u_{i+1} G(b)$, which gives that $u_i g^{'}(a) \ge u_{i+1} g^{'}(b)$. Hence, ($ii$) follows from Proposition H.2 of Marshall et al. (2011) □

The following counterexample shows that if $g(\cdot)$ is increasing and convex and $\mathbf{u} \in \mathcal{D}^+$, then $\phi(\mathbf{x})$ may not be Schur-convex or Schur-concave.

Counterexample 3.1 *Let $g(x) = e^x$, $\mathbf{x} = (30, 8, 2)$ and $\mathbf{y} = (18, 12, 10)$. So, clearly $g(x)$ is increasing and convex and $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$. Now, if $\mathbf{u} = (1, 2, 3) \in \mathcal{D}^+$ is taken, then it can be easily checked that*

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = 1.068640854 \times 10^{13} > 0,$$

giving $\phi(\mathbf{x}) > \phi(\mathbf{y})$. *Again, if* $\mathbf{x} = (4, 3, 1)$ *and* $\mathbf{y} = (3, 3, 2)$ *is taken then, for* $\mathbf{u} = (1, 2, 30) \in \mathcal{D}^+$ *and for the same function* $g(\cdot)$, *it can be easily checked that, although* $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$,

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = -105.610615 < 0,$$

giving $\phi(\mathbf{x}) < \phi(\mathbf{y})$. *So,* $\phi(\mathbf{x})$ *is neither Schur-convex nor Schur-concave.* □

That nothing can be said about the Schur-convexity of $\phi(\mathbf{x})$ when $g(\cdot)$ is increasing and concave, and $\mathbf{u} \in \mathcal{D}_+$, is shown in the next counterexample.

Counterexample 3.2 *For* $\mathbf{x} = (30, 8, 2)$, $\mathbf{y} = (18, 12, 10)$ *and* $\mathbf{u} = (3, 2, 1) \in \mathcal{D}_+$, *if* $g(x) = \ln x$ *is taken, which is increasing and concave, then*

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = -0.887891257 < 0,$$

giving $\phi(\mathbf{x}) < \phi(\mathbf{y})$. *Again, for* $\mathbf{x} = (4, 3, 1)$, $\mathbf{y} = (3, 3, 2)$ *and* $\mathbf{u} = (30, 2, 1) \in \mathcal{D}_+$ *and, for the same function* $g(\cdot)$, *it can be seen that*

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = 7.937314993 > 0,$$

satisfying the claim. □

The counterexample given below shows that $\phi(\mathbf{x})$ is neither Schur-convex nor Schur-concave if the function $g(\cdot)$ is decreasing and convex and $\mathbf{u} \in \mathcal{D}_+$.

Counterexample 3.3 *Let* $g(x) = e^{-x}$, *which is decreasing and convex, and* $\mathbf{x} = (4, 3, 1)$ *and* $\mathbf{y} = (3, 3, 2)$. *Now, if* $\mathbf{u} = (3, 2, 1) \in \mathcal{D}_+$ *is taken, then it can be easily verified that*

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = 0.138129869 > 0,$$

giving $\phi(\mathbf{x}) > \phi(\mathbf{y})$. *Again, if we take* $\mathbf{x} = (3, 2, 1)$ *and* $\mathbf{y} = (2, 2, 2)$ *then, for* $\mathbf{u} = (26, 2, 1) \in \mathcal{D}_+$ *and for the same function* $g(\cdot)$, *it can be easily checked that, although* $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$,

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = -1.991709429 < 0,$$

giving $\phi(\mathbf{x}) < \phi(\mathbf{y})$. *So,* $\phi(\mathbf{x})$ *is neither Schur-convex nor Schur-concave.* □

The following counterexample shows that if $g(\cdot)$ is decreasing and concave and $\mathbf{u} \in \mathcal{D}^+$, then $\phi(\mathbf{x})$ is neither Schur-convex nor Schur-concave function.

Counterexample 3.4 *Let* $g(x) = 1 - e^{-9x^{-0.4}}$, *which is decreasing and concave for all* $x \in [0, 10]$. *Now, if we take* $\mathbf{x} = (8, 4, 3)$, $\mathbf{y} = (7, 4, 4)$ *and* $\mathbf{u} = (10, 10.2, 10.4) \in \mathcal{D}^+$, *then it can also be checked that although* $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$,

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = -0.0108009 < 0,$$

giving $\phi(\mathbf{x}) < \phi(\mathbf{y})$. *Again, for the same function* $g(\cdot)$ *and for same* $\mathbf{x}$, $\mathbf{y}$, *if* $\mathbf{u} = (1, 20, 30) \in \mathcal{D}^+$ *is taken then*

$$\sum_{i=1}^{3} u_i g(x_i) - \sum_{i=1}^{3} u_i g(y_i) = 0.0759684 > 0,$$

giving $\phi(\mathbf{x}) > \phi(\mathbf{y})$. *So,* $\phi(\mathbf{x})$ *is neither Schur-convex nor Schur-concave.* □

From Counterexample 3.3 it is clear that the parenthetical statement of Proposition H.2.b in Marshall et al. (2011, p. 133), is not correct. We also observe that the parenthetical statement of Proposition H.2.b with $u_i \equiv 1$ contradicts Proposition C.1 of Marshall et al. (2011, p. 92). The observations from the above results and counterexamples are reported in the following table.

	g is increasing		g is decreasing	
	$\mathbf{u} \in \mathcal{D}_+$	$\mathbf{u} \in \mathcal{D}^+$	$\mathbf{u} \in \mathcal{D}_+$	$\mathbf{u} \in \mathcal{D}^+$
g is convex	s-convex	Inconclusive	Inconclusive	s-convex
g is concave	Inconclusive	s-concave	s-concave	Inconclusive

Below we give a lemma without proof which will be used to prove Lemma 3.3 which in turn will be used to prove the upcoming theorem.

Lemma 3.2 *For all* $x, t \geq 0$, *let* $\phi(tx)$ *be decreasing in* x. *Then* $g(x) = x\phi(tx)$ *is convex if and only if, for all* $x, t \geq 0$,

$$2\phi_1(tx) + tx\phi_1'(tx) < 0,$$

where $\phi_1(tx) = -\phi'(tx) > 0$ *and* $\phi'(z) = \frac{d}{dz}\phi(z)$. □

The following lemma is used to prove Theorem 3.2.

Lemma 3.3 *For* $y > 0$, $g(y) = y\left(e^{yx} - 1\right)^{-1}$ *is decreasing and convex in* y.

Proof: That $g(y)$ is decreasing can be seen by differentiation whereas the convexity of $g(y)$ can be proved by some algebra with the help of Lemma 3.2. □

The following theorem shows that, if $\boldsymbol{\lambda}$ majorizes $\boldsymbol{\delta}$, then $X_{n:n}$ will be greater than $Y_{n:n}$ in reversed hazard rate ordering.

Theorem 3.2 *If* X_i, Y_i *follow generalized exponential distributions with parameters* (λ_i, θ_i) *and* (δ_i, θ_i) *respectively, for* $i = 1, 2, \ldots, n$, *then* $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\delta}$ *implies* $X_{n:n} \geq_{rh} Y_{n:n}$.

Proof: Let $\overline{r}_X(\cdot;\boldsymbol{\lambda},\boldsymbol{\theta})$ and $\overline{r}_Y(\cdot;\boldsymbol{\delta},\boldsymbol{\theta})$ be the reversed hazard rate functions of $X_{n:n}$ and $Y_{n:n}$ respectively. Then, clearly

$$\begin{aligned}\overline{r}_X(x;\boldsymbol{\lambda},\boldsymbol{\theta}) &= \sum_{i=1}^n \frac{\theta_i\lambda_i e^{-\lambda_i x}}{1-e^{-\lambda_i x}} = \sum_{i=1}^n \theta_i g(\lambda_i),\\ \overline{r}_Y(x;\boldsymbol{\delta},\boldsymbol{\theta}) &= \sum_{i=1}^n \frac{\theta_i\delta_i e^{-\delta_i x}}{1-e^{-\delta_i x}} = \sum_{i=1}^n \theta_i g(\delta_i), say,\end{aligned}$$

where $g(y)=\frac{y}{e^{yx}-1}$ for all $y>0$. Now, it is enough to prove that $\overline{r}_X(x;\boldsymbol{\lambda},\boldsymbol{\theta})$ is Schur-convex, which follows from Lemma 3.3 and Theorem 3.1 $b(ii)$.

Remark 3.1 *Theorem 3.2 improves Theorem 3.2 of Dykstra et al. (1997) in the sense that the later can be obtained from the former by taking $\theta_1=\theta_2=\ldots=\theta_n=1$.* □

Now the question arises whether Theorem 3.2 can be strengthened further by replacing rh order between $X_{n:n}$ and $Y_{n:n}$ by lr order. Although, for $n=3$, the following counterexample gives a negative answer to it, Theorem 3.3 extends Theorem 3.2 to likelihood ratio ordering for $n=2$.

Counterexample 3.5 *For $n=3$, let $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3)=(8,2,0.1)\in\mathcal{D}_+$, $\boldsymbol{\delta}=(\delta_1,\delta_2,\delta_3)=(5,5,0.1)\in\mathcal{D}_+$ and $\boldsymbol{\theta}=(\theta_1,\theta_2,\theta_3)=(1,2,3)\in\mathcal{D}^+$. Clearly, $\boldsymbol{\lambda}\overset{m}{\succeq}\boldsymbol{\delta}$. Now*

$$F_{3:3}(x)=\left(1-e^{-8x}\right)\left(1-e^{-2x}\right)^2\left(1-e^{-0.1x}\right)^3$$

and

$$G_{3:3}(x)=\left(1-e^{-5x}\right)\left(1-e^{-5x}\right)^2\left(1-e^{-0.1x}\right)^3.$$

Let $f_{n:n}(\cdot)$ and $g_{n:n}(\cdot)$ be the density functions of $X_{n:n}$ and $Y_{n:n}$ respectively. Then, by writing $a(y)=\frac{f_{3:3}(x)}{g_{3:3}(x)}$, where $x=-\ln y$, $0<y<1$, we see from Figure 3.1 that $a(y)$ is not monotone. Thus, there is no lr ordering between $X_{n:n}$ and $Y_{n:n}$. □

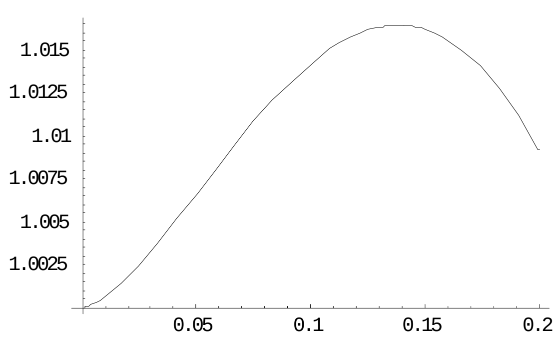

Figure 3.1: Graph of $a(y)$

Theorem 3.3 *Let X_i, Y_i follow generalized exponential distributions with parameters (λ_i, θ_i) and (δ_i, θ_i) respectively, for $i = 1, 2$, then $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\delta}$ implies $X_{2:2} \geq_{lr} Y_{2:2}$.*

Proof: Let $F_{2:2}(\cdot)$ and $G_{2:2}(\cdot)$ be the distribution functions of $X_{2:2}$ and $Y_{2:2}$ respectively. Then

$$F_{2:2}(x) = \prod_{i=1}^{2} \left(1 - e^{-\lambda_i x}\right)^{\theta_i}$$

and

$$G_{2:2}(x) = \prod_{i=1}^{n} \left(1 - e^{-\delta_i x}\right)^{\theta_i}.$$

If $f_{2:2}(x)$ and $g_{2:2}(x)$ be the density functions of $X_{2:2}$ and $Y_{2:2}$ respectively, then

$$\frac{f_{2:2}(x)}{g_{2:2}(x)} = \frac{\frac{\lambda_1\theta_1}{e^{\lambda_1 x}-1} + \frac{\lambda_2\theta_2}{e^{\lambda_2 x}-1}}{\frac{\delta_1\theta_1}{e^{\delta_1 x}-1} + \frac{\delta_2\theta_2}{e^{\delta_2 x}-1}} \frac{F_{2:2}(x)}{G_{2:2}(x)}.$$

As $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\delta}$, by Theorem 3.2 it can be written that $\frac{F_{2:2}(x)}{G_{2:2}(x)}$ is increasing in $x \geq 0$. So, we have only to show that

$$g(x) = \frac{\frac{\lambda_1\theta_1}{e^{\lambda_1 x}-1} + \frac{\lambda_2\theta_2}{e^{\lambda_2 x}-1}}{\frac{\delta_1\theta_1}{e^{\delta_1 x}-1} + \frac{\delta_2\theta_2}{e^{\delta_2 x}-1}} = \frac{\theta_1 u(\lambda_1 x) + \theta_2 u(\lambda_2 x)}{\theta_1 u(\delta_1 x) + \theta_2 u(\delta_2 x)}$$

is increasing in x, where $u(x) = \frac{x}{e^x - 1}$. Now, $u^{'}(x) = \frac{u(x)v(x)}{x}$ (say) where $v(x) = \frac{e^x - 1 - xe^x}{e^x - 1}$. It can be easily shown that both $u(x)$ and $v(x)$ are decreasing in x. Now, as $g^{'}(x)$, the differentiation of $g(x)$ with respect to x, gives

$$\begin{aligned} g^{'}(x) \overset{sign}{=} & \left(\theta_1 u(\delta_1 x) + \theta_2 u(\delta_2 x)\right)\left(\theta_1 u(\lambda_1 x)v(\lambda_1 x) + \theta_2 u(\lambda_2 x)v(\lambda_2 x)\right) - \\ & \left(\theta_1 u(\lambda_1 x) + \theta_2 u(\lambda_2 x)\right)\left(\theta_1 u(\delta_1 x)v(\delta_1 x) + \theta_2 u(\delta_2 x)v(\delta_2 x)\right), \end{aligned}$$

it is clear that $g(x)$ is increasing in $x \geq 0$ if

$$\Psi(\lambda_1, \lambda_2) = \frac{\theta_1 u(\lambda_1 x)v(\lambda_1 x) + \theta_2 u(\lambda_2 x)v(\lambda_2 x)}{\theta_1 u(\lambda_1 x) + \theta_2 u(\lambda_2 x)}$$

is schur convex in (λ_1, λ_2). Now, differentiating $\Psi(\lambda_1, \lambda_2)$ with respect to λ_1 and λ_2, we get

$$\frac{\partial \Psi}{\partial \lambda_1} \overset{sign}{=} \theta_1 x \left[\theta_2 u^{'}(\lambda_1 x)u(\lambda_2 x)(v(\lambda_1 x) - v(\lambda_2 x)) + u(\lambda_1 x)v^{'}(\lambda_1 x)(\theta_1 u(\lambda_1 x) + \theta_2 u(\lambda_2 x))\right]$$

and

$$\frac{\partial \Psi}{\partial \lambda_2} \overset{sign}{=} \theta_2 x \left[\theta_1 u^{'}(\lambda_2 x)u(\lambda_1 x)(v(\lambda_2 x) - v(\lambda_1 x)) + u(\lambda_2 x)v^{'}(\lambda_2 x)(\theta_1 u(\lambda_1 x) + \theta_2 u(\lambda_2 x))\right].$$

Assuming $w(x) = u(x)v^{'}(x)$, we get

$$\begin{aligned} \frac{\partial \Psi}{\partial \lambda_1} - \frac{\partial \Psi}{\partial \lambda_2} \overset{sign}{=} & \; x \left[(\theta_1\theta_2(v(\lambda_1 x) - v(\lambda_2 x))(u^{'}(\lambda_1 x)u(\lambda_2 x) + u^{'}(\lambda_2 x)u(\lambda_1 x))) \right. \\ & \left. + \; ((\theta_1 u(\lambda_1 x) + \theta_2 u(\lambda_2 x))(\theta_1 w(\lambda_1 x) - \theta_2 w(\lambda_2 x)))\right]. \end{aligned} \tag{3.2}$$

Again, as $w(x) = u(x)v^{'}(x)$ and $v(x)$ is decreasing in x, then for all $x \geq 0$, $w(x) < 0$. Again, by Lemma (3.3) of Torrado and Kochar (2014), $w(x)$ is increasing in x. So, as $\lambda_1 \geq \lambda_2$ and

a) $v(x)$ is decreasing in x, $(v(\lambda_1 x) - v(\lambda_2 x)) < 0$,

b) $u(x)$ is decreasing in x, $u^{'}(\lambda_1 x), u^{'}(\lambda_2 x) < 0$, and

c) $w(x)$ is increasing and negative for all $x \geq 0$, $\theta_1 w(\lambda_1 x) \leq \theta_2 w(\lambda_2 x)$ for all $\theta_1 \leq \theta_2$ and $x \geq 0$.

Therefore, from (3.2) it is clear that $\frac{\partial \Psi}{\partial \lambda_1} - \frac{\partial \Psi}{\partial \lambda_2} > 0$, proving $\Psi(\lambda_1, \lambda_2)$ is Schur-convex by Theorem A3 of Marshall et al.

Remark 3.2 *The above theorem improves Theorem 3.1 of Dysktra et al. (1997) in the sense that the later can be obtained from the former by taking* $\theta_1 = \theta_2 = 1$.

Below we also see that there exists lr order between $X_{n:n}$ and $Y_{n:n}$ if Y_i is a random variable following generalized exponential distribution with parameter $\left(\overline{\lambda}, \theta_i\right)$, $i = 1, 2, \ldots, n$, where $\overline{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i$.

Theorem 3.4 *Let* X_i, Y_i *follow generalized exponential distributions with parameters* (λ_i, θ_i) *and* $\left(\overline{\lambda}, \theta_i\right)$ *respectively, for* $i = 1, 2, \ldots, n$. *Then* $X_{n:n} \geq_{lr} Y_{n:n}$.

Proof: Let $F_{n:n}(\cdot)$ and $G_{n:n}(\cdot)$ be the distribution functions of $X_{n:n}$ and $Y_{n:n}$ respectively. Then

$$\log F_{n:n}(x) = \sum_{i=1}^{n} \theta_i \log \left(1 - e^{-\lambda_i x}\right)$$

and

$$\log G_{n:n}(x) = \sum_{i=1}^{n} \theta_i \log \left(1 - e^{-\overline{\lambda} x}\right),$$

which, on differentiation with respect to x, gives

$$f_{n:n}(x) = F_{n:n}(x) \sum_{i=1}^{n} \frac{\theta_i \lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}}$$

and

$$g_{n:n}(x) = G_{n:n}(x) \sum_{i=1}^{n} \frac{\theta_i \overline{\lambda} e^{-\overline{\lambda} x}}{1 - e^{-\overline{\lambda} x}},$$

where $f_{n:n}(\cdot)$ and $g_{n:n}(\cdot)$ are the density functions of $X_{n:n}$ and $Y_{n:n}$ respectively. So, to show that $X_{n:n} \geq_{lr} Y_{n:n}$, we have to show that

$$\frac{f_{n:n}(x)}{g_{n:n}(x)} = \frac{1}{n\overline{\theta}\,\overline{\lambda}} \sum_{i=1}^{n} \frac{\theta_i \lambda_i \left(e^{\overline{\lambda} x} - 1\right)}{\left(e^{\lambda_i x} - 1\right)} \frac{F_{n:n}(x)}{G_{n:n}(x)}, \tag{3.3}$$

where $\overline{\theta} = \frac{1}{n} \sum_{i=1}^{n} \theta_i$, is increasing in $x \geq 0$. Now, from Theorem 3.2 and (2.1), and noting the fact that $(\lambda_1, \lambda_2 \ldots, \lambda_n) \stackrel{m}{\succeq} \left(\overline{\lambda}, \overline{\lambda} \ldots, \overline{\lambda}\right)$, it is clear that $\frac{F_{n:n}(x)}{G_{n:n}(x)}$ is increasing in $x \geq 0$. So, from (3.2), it is only required to show that

$$g(x) = \sum_{i=1}^{n} \frac{\theta_i \lambda_i \left(e^{\overline{\lambda} x} - 1\right)}{\left(e^{\lambda_i x} - 1\right)}$$

is increasing in $x \geq 0$. Differentiating $g(x)$ with respect to x, we have

$$g^{'}(x) = \overline{\lambda} e^{\overline{\lambda} x} \sum_{i=1}^{n} \frac{\theta_i \lambda_i}{\left(e^{\lambda_i x} - 1\right)} - \left(e^{\overline{\lambda} x} - 1\right) \sum_{i=1}^{n} \frac{\theta_i \lambda_i^2 e^{\lambda_i x}}{\left(e^{\lambda_i x} - 1\right)^2}. \tag{3.4}$$

It can be shown that each of $\frac{\lambda_i^2 e^{-\lambda_i x}}{\left(1-e^{-\lambda_i x}\right)^2}$ and $\frac{1}{\lambda_i}\left(1 - e^{-\lambda_i x}\right)$ is decreasing in λ_i. Thus, for $\boldsymbol{\theta} \in \mathcal{D}^+$ and $\boldsymbol{\lambda} \in \mathcal{D}_+$, we have, by using Equation (1.5) of Mitrinović et al. (1993, p. 240), that

$$\sum_{i=1}^{n} \frac{\theta_i \lambda_i}{e^{\lambda_i x} - 1} \geq \frac{1}{n} \sum_{i=1}^{n} \frac{\theta_i \lambda_i^2 e^{-\lambda_i x}}{\left(1 - e^{-\lambda_i x}\right)^2} \sum_{i=1}^{n} \frac{\left(1 - e^{-\lambda_i x}\right)}{\lambda_i}.$$

Thus, $g(x)$ is increasing in x, if for all $x \geq 0$,

$$\frac{\overline{\lambda}}{n} \sum_{i=1}^{n} \frac{1 - e^{-\lambda_i x}}{\lambda_i} - \left(1 - e^{-\overline{\lambda} x}\right) \geq 0,$$

which holds by judiciously using $AM - GM$ inequality. Hence, the result follows.

Remark 3.3 *The above theorem improves Theorem 2.1 (b) of Dykstra et al. (1997) in the sense that the later can be obtained from the former by taking $\theta_1 = \theta_2 = \ldots = \theta_n = 1$ and by noticing the fact that lr ordering implies fr ordering.*

We have shown through counterexamples that likelihood ratio ordering between $X_{n:n}$ and $Y_{n:n}$ with heterogeneous generalized exponential components is not possible for any positive integer n. Next theorem shows that a similar result still holds for multiple-outlier generalized exponential model.

Theorem 3.5 *Let $X_1, X_2, \ldots, X_n$ be independent random variables following the multiple-outlier generalized exponential model such that $X_i \sim GE\left(\theta_i, \lambda_1\right)$ for $i = 1, 2, \ldots, n_1$ and $X_j \sim GE\left(\phi_j, \lambda_2\right)$ for $j = 1, 2, \ldots, n_2, n_1 + n_2 = n$ with $\lambda_1 > \lambda_2$ and $\theta_1 < \theta_2 < \ldots < \theta_{n_1} < \phi_1 < \phi_2 < \ldots < \phi_{n_2}$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables following multiple-outlier generalized exponential model such that $Y_i \sim GE\left(\theta_i, \delta_1\right)$ for $i = 1, 2, \ldots, n_1$ and $Y_j \sim GE\left(\phi_j, \delta_2\right)$ for $j = 1, 2, \ldots, n_2$, $n_1 + n_2 = n$ with $\delta_1 > \delta_2$ and same conditions on θ_i's and ϕ_j's as for the random variables X_i's. If $\sum_{i=1}^{n_1} \theta_i \leq \sum_{i=1}^{n_2} \phi_i$, then $(\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{n_1}, \underbrace{\lambda_2, \lambda_2, \ldots, \lambda_2}_{n_2}) \overset{m}{\succeq} (\underbrace{\delta_1, \delta_1, \ldots, \delta_1}_{n_1}, \underbrace{\delta_2, \delta_2, \ldots, \delta_2}_{n_2})$ implies $X_{n:n} \geq_{lr} Y_{n:n}$.*

Proof: Let $\sum_{i=1}^{n_1} \theta_i = \theta$ and $\sum_{i=1}^{n_2} \phi_i = \phi$. As defined before, distribution functions of $X_{n:n}$ and $Y_{n:n}$ are given as

$$F_{n:n}(x) = \prod_{i=1}^{n_1} \left(1 - e^{-\lambda_1 x}\right)^{\theta_i} \prod_{j=1}^{n_2} \left(1 - e^{-\lambda_2 x}\right)^{\phi_j} = \left(1 - e^{-\lambda_1 x}\right)^{\theta} \left(1 - e^{-\lambda_2 x}\right)^{\phi}$$

and

$$G_{n:n}(x) = \prod_{i=1}^{n_1} \left(1 - e^{-\delta_1 x}\right)^{\theta_i} \prod_{j=1}^{n_2} \left(1 - e^{-\delta_2 x}\right)^{\phi_j} = \left(1 - e^{-\delta_1 x}\right)^{\theta} \left(1 - e^{-\delta_2 x}\right)^{\phi}$$

with

$$\frac{f_{n:n}(x)}{g_{n:n}(x)} = \frac{\frac{\lambda_1\theta}{e^{\lambda_1 x}-1} + \frac{\lambda_2\phi}{e^{\lambda_2 x}-1}}{\frac{\delta_1\theta}{e^{\delta_1 x}-1} + \frac{\delta_2\phi}{e^{\delta_2 x}-1}} \frac{F_{n:n}(x)}{G_{n:n}(x)}.$$

Using the same logic and same symbols as used in Theorem (3.3), to show that $X_{n:n} \geq_{lr} Y_{n:n}$, we have to show that

$$g(x) = \frac{\frac{\lambda_1\theta}{e^{\lambda_1 x}-1} + \frac{\lambda_2\phi}{e^{\lambda_2 x}-1}}{\frac{\delta_1\theta}{e^{\delta_1 x}-1} + \frac{\delta_2\phi}{e^{\delta_2 x}-1}} = \frac{\theta u(\lambda_1 x) + \phi u(\lambda_2 x)}{\theta u(\delta_1 x) + \phi u(\delta_2 x)}$$

is increasing in x, where $u(x) = \frac{x}{e^x-1}$. Now taking $u^{'}(x) = \frac{u(x)v(x)}{x}$ where $v(x) = \frac{e^x-1-xe^x}{e^x-1}$, it can be easily shown that both $u(x)$ and $v(x)$ are decreasing in x. Next, differentiating $g(x)$ as before

$$g^{'}(x) \overset{sign}{=} (\theta u(\delta_1 x) + \phi u(\delta_2 x))\,(\theta u(\lambda_1 x)v(\lambda_1 x) + \phi u(\lambda_2 x)v(\lambda_2 x)) - (\theta u(\lambda_1 x) + \phi u(\lambda_2 x))\,(\theta u(\delta_1 x)v(\delta_1 x) + \phi u(\delta_2 x)v(\delta_2 x)\,,$$

and to show that the above expression is greater than zero, we have only to prove that the function

$$\Psi(\lambda_1, \lambda_2) = \frac{\theta u(\lambda_1 x)v(\lambda_1 x) + \phi u(\lambda_2 x)v(\lambda_2 x)}{\theta u(\lambda_1 x) + \phi u(\lambda_2 x)}$$

is schur convex in (λ_1, λ_2). Assuming $w(x) = u(x)v^{'}(x)$, and proceeding in the similar way as before we get

$$\frac{\partial\Psi}{\partial\lambda_1} - \frac{\partial\Psi}{\partial\lambda_2} \overset{sign}{=} x\left[\left(\theta\phi(v(\lambda_1 x) - v(\lambda_2 x))(u^{'}(\lambda_1 x)u(\lambda_2 x) + u^{'}(\lambda_2 x)u(\lambda_1 x))\right) + ((\theta u(\lambda_1 x) + \phi u(\lambda_2 x))(\theta w(\lambda_1 x) - \phi w(\lambda_2 x)))\right]. \tag{3.5}$$

Hence, following the same logic as in Theorem 3.3 $\left(\frac{\partial\Psi}{\partial\lambda_1} - \frac{\partial\Psi}{\partial\lambda_2}\right) > 0$ as $\theta \leq \phi$, proving the result.

Remark 3.4 *The above theorem improves Theorem 3.5 of Zhao and Balakrishnan (2012) in the sense that the later can be obtained from the former by taking* $\theta_1 = \theta_2 = \ldots = \theta_{n_1} = \phi_1 = \phi_2 = \ldots = \phi_{n_2} = 1$ *and assuming* $n_1 \leq n_2$.

4 Conclusions

Although the concept of majorization started in order to compare the income inequalities, nowadays one can find application of majorization in different branches of economics, reliability, engineering and many others. In this paper, we compare the lives of two parallel systems formed by components having heterogeneous generalized exponentially distributed lifetimes. It is shown that if the vectors of parameters of the underlying distributions are ordered in the sense of majorization, then the life of one parallel system will be more than that of the other in reversed hazard rate order. We also show with the help of counterexample that this result cannot be extended to likelihood ratio (lr) order. However, we have shown that, under certain restriction, the result can be extended to the lr order. In the process of development of these two main results of the paper, we have corrected a mistake in the book by Marshall et al. (2011), and also proved some new results which, we are sure, will enrich the theory of majorization up to certain extent.

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Stochastic Comparisons of Parallel Systems of Heterogeneous Generalized Exponential Components

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Abstract:

Let $X_1, X_2, \ldots, X_n$ (resp. $Y_1, Y_2, \ldots, Y_n$) be independent random variables such that X_i (resp. Y_i) follows generalized exponential distribution with shape parameter θ_i and scale parameter λ_i (resp. δ_i), $i = 1, 2, \ldots, n$. Here it is shown that if $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ majorizes $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$ then $X_{n:n}$ will be greater than $Y_{n:n}$ in reversed hazard rate ordering. That no relation exists between $X_{n:n}$ and $Y_{n:n}$, under same condition, in terms of likelihood ratio ordering has also been shown. It is also shown that, if Y_i follows generalized exponential distribution with parameters $(\overline{\lambda}, \theta_i)$, where $\overline{\lambda}$ is the mean of all λ_i's, $i = 1 \ldots n$, then $X_{n:n}$ is greater than $Y_{n:n}$ in likelihood ratio ordering. In this context, an error in Marshall, Olkin and Arnold [Inequalities: Theory of Majorization and Its applications (2011)] has been corrected, and some new results on majorization have been developed.

Key Words/Phrases: Hazard rate function, majorization, reversed hazard rate function, Schur-convex and Schur-concave functions, stochastic orders.

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