

Chaotic Synchronization and the Global Economy

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Abstract

A brief introductory article on the role of chaotic synchronization in the context of complex economic systems. The basic framework developed by the late Richard Goodwin in his book, *Chaotic Economic Dynamics*, of 1990 has been extended to massively complex dynamical systems of chaotic elements. Recent experimental results and speculative applications to global economic systems are presented.¹

Keywords

Rössler attractor, chaotic synchronization, lattice dynamics, complex economic dynamics, complex dynamical system

Introduction

In the long history of nonlinear dynamics from Lord Rayleigh (1877) to the present, ‘forced vibrations’ has been a major thread.² The synchronization, or entrainment, of two oscillators in frequency and in phase, discovered famously by Christiaan Huyghens in 1665, is an aspect of this thread with many important applications in the sciences: physical, biological and social.³ With the advent of the modern qualitative theory of dynamical systems in the 1960s, an intuitive geometric theory emerged in which the phase entrainment of an oscillator may be anticipated.⁴ As this theory applied to chaotic attractors as well as to periodic ones, the synchronization of chaotic attractors was predicted and verified experimentally in the 1980.⁵ And thus was born the field of ‘chaos synchronization’. With extraordinary prescience, the late Richard Goodwin applied these ideas to complex economic systems in his classic of 1990, *Chaotic Economic Dynamics*.

Here we extend his ideas to large scale lattices of Rössler systems and present preliminary results of computer simulations in which islands (anti-nodes) of synchronization are constrained by nodal lines, as in vibrating membranes. These results, reminiscent of the dynamics of cardiac tissue⁶, are highly suggestive for economic systems, and we end with some speculations on these implications.

The evolution of lattice dynamics may be summarized in this brief chronology.

- 1692, physics of the hanging chain (Bernoulli)
- 1826, spatial economics (Von Thünen)
- 1930, biological morphogenesis (Fisher, Kolmogorov, Turing)
- 1970, chemical waves (Belousov–Zhabotinsky)
- 1990, economic networks (Goodwin)

Glossary

We begin with a brief glossary of terms.

‘Complexity’ is an ambiguous term referring to some of these ideas: chaos theory, cellular automata, genetic algorithms, networks, system dynamics and complex systems.⁷ A ‘dynamical system’ is either:⁸

- a ‘flow’, defined by a vectorfield (a system of autonomous ordinary differential equations) defined on a state space, S ,
- a ‘cascade’, defined by a reversible endomorphism of S , or
- an ‘iteration’, defined by an arbitrary endomorphism of S .

The state space in general is a finite-dimensional differentiable manifold, but here we will consider only the special case, $S \subset R^n$ (cartesian n -space).

A network consists of nodes and directed or undirected links. A complex dynamical system comprises a directed network together with data: every node contains a dynamical scheme (a dynamical system with control parameters) and every directed link connects states at its tail to controls at its head.

Rössler Systems

Here we will be concerned with complex dynamical systems with flow schemes at each node. Especially, the case

in which the same scheme—for example, the Rössler scheme—is attached to each node:⁹

$$u' = au + v \quad (1)$$

$$v' = -u - z \quad (2)$$

$$z' = b + (v - c)z \quad (3)$$

Following Richard Goodwin, we might set $a = b = 0.2$, so we have a three-dimensional flow scheme at each node with a single control parameter, c . The original value for the familiar Rössler attractor is $c = 5.7$. If the control parameter, c , increases from zero to 50 or so, a bifurcation sequence is observed that resembles somewhat the well-known behaviour of the logistic family of one-dimensional maps. Some special cases are shown in Figure 1.

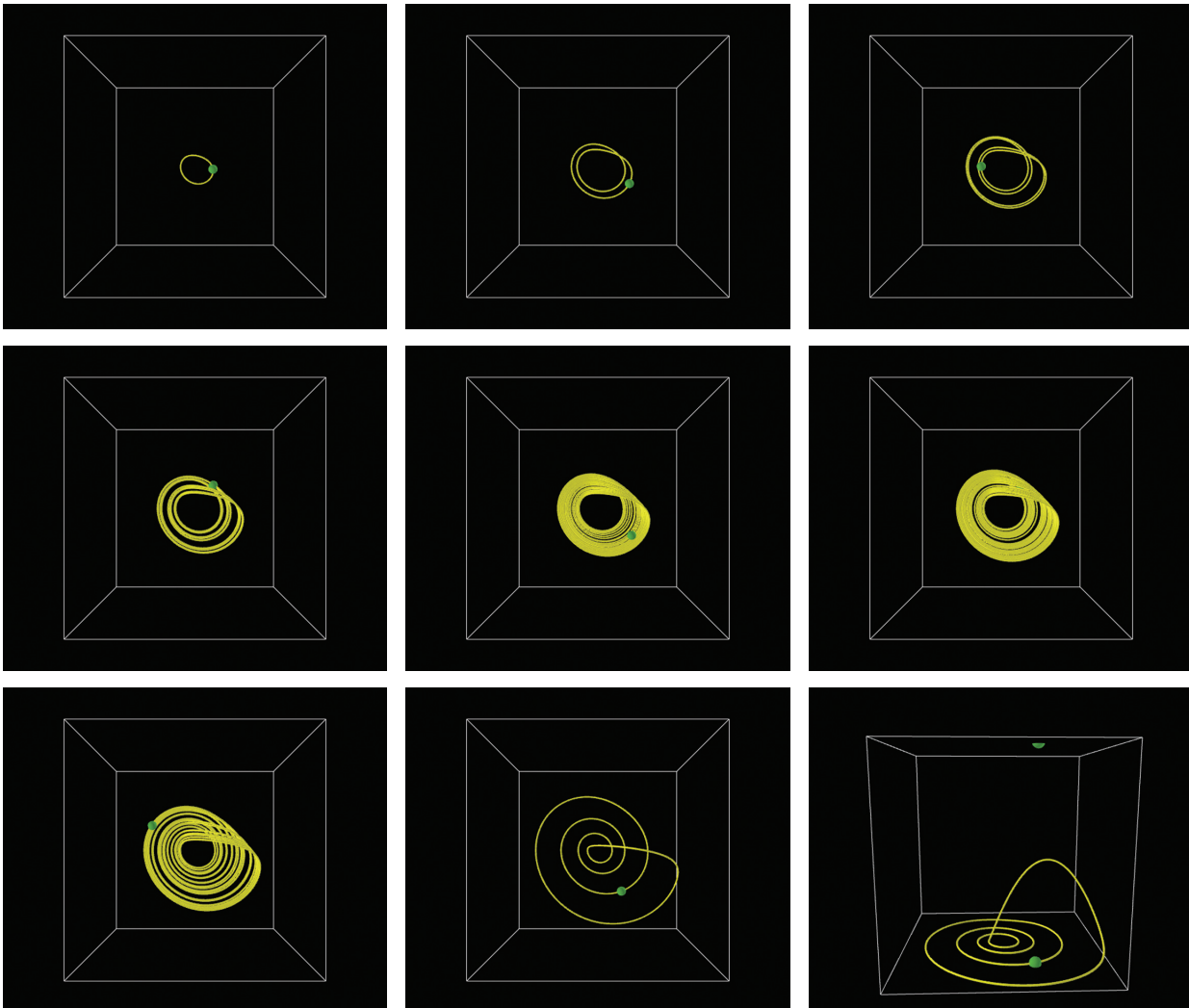


Figure 1. Dependence of Rössler attractor upon control c . The successive values of c are 1.6, 3.0, 3.9, 4.2, 4.5, 5.7 (the standard value) and 7.0. This latter case belongs to a periodic window. The views of these eight cases are looking down from the positive z axis. The ninth image is again $c = 7.0$, but seen from the positive u axis.

Alternatively, we may fix $a = 0.2$, $c = 5.7$ and allow b be controlled from outside. In a forced system, a directed link will determine the value of b at its head from the state (u, v, z) at its tail. And our simplest networks will consist of only one or two nodes, each containing a Rössler system with identical values of the parameters, a and c , while the value of b is controlled.

In the case of one node, the Rössler scheme will have a single attractor, either fixed, periodic, or chaotic, depending upon the value of the control parameter, c . The bifurcation sequence for this scheme is well known: Increasing c from zero, we encounter a single attractor, first a point, then a periodic cycle going once around the central point, the so-called unit attractor. This cycle then undergoes a period-doubling sequence converging to chaos and exhibiting for some intervals, a periodic window. The bifurcation sequence of this scheme as b is varied also includes period doubling routes to chaos, as well as periodic windows, as shown in Figure 2.¹⁰

In the case of two nodes and one link, we have a forced Rössler system. These systems may exhibit useful properties of phase entrainment. In the case of two nodes and two links, we have a coupled Rössler system. These systems also exhibit phase entrainment.

Goodwin has employed the Rössler scheme to model economic units such as factories. In this context, the variables u and v may represent wages and profits, respectively, which cycle roughly periodically like predators and prey, while the variable z represents an interactive policy factor.¹¹ Meanwhile, the control parameter, c , may be manipulated by management or other exogenous factors, changing the shape and character of the unique attractor.

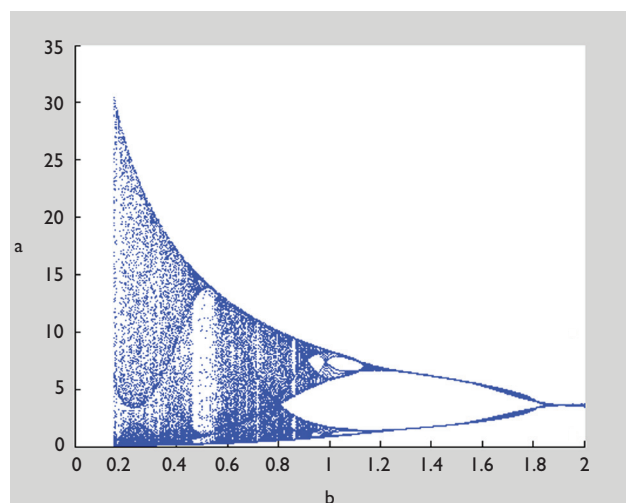


Figure 2. Dependence of Rössler attractor upon control b
Source: Developed by the author.

Chaos with Periodic Forcing

Consider now an application in which one Rössler scheme, representing a bank let's say, is subject to periodic forcing by an exogenous institution such as a government or central bank. If the forcing is applied subtly, it will approximate the near-periodicity of the chaotic attractor and seek to stabilize its phase by pushing up in the z -direction as the controlled unit approaches its z -peak, advancing its phase, and pushing down as the controlled unit passes its peak, retarding its phase. The result is better periodic behaviour of the forced system. This is the essence of the geometric theory of phase entrainment. Goodwin has considered the case in which a Rössler scheme is subject to periodic exogenous forcing with chaotic results.¹² This master/slave situation may be represented in a conspiracy theory, in which some power group, perhaps imaginary like the Bavarian Illuminati, is forcing up the interest rates or market prices by greedy manipulation. Note that the forcing term effectively alters the control parameter b , thus driving the system over the cliff of the successive bifurcations shown in Figure 2.

Chaos with Chaotic Forcing

We now consider a Rössler scheme forced by another Rössler system. Recall that the Rössler attractor is roughly a smooth oscillation in the (u, v) plane, accompanied by randomly occurring spikes in the z direction. This behaviour is similar to that of a Hodgkin—Huxley neuron model, with periodic equal spiking replaced by a rough equivalent, which is chaotic both in the timing and the strength of the spikes. With no connection between two Rössler systems, the two (u, v) nearly periodic oscillators will be uncorrelated and the spikes as well. But with weak coupling from the master z -spike to the b -parameter of the slave system, partial synchronization occurs.

In Figure 3 we show the parallel trajectories of the master system (orange) and the slave (yellow), drawn in 3D by a Net Logo simulation. In Figure 4 we can see the time variation of the z coordinate of the master (green) along with that of the slave (blue). Note that stronger spikes are frequently in sync, while weaker ones are not.

We may obtain a rough measure of the synchronization of the spikes in this figure as follows. Let us set a threshold value, 1.0, to define a spike. That is, when the w -coordinate of a system exceeds 1.0, we say it is spiking. Now during a simulation run, add together all the time intervals of the

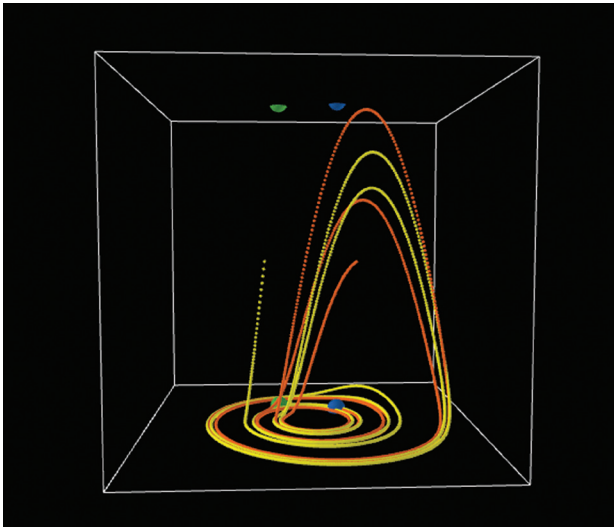


Figure 3. Master (yellow) and slave (orange)

master system that it is spiking, s_0 , and all the time intervals, sb , that both master and slave are spiking and express sb/s_0 as a percentage. This approximates the correlation of the spikes of the two systems. It takes account of the slave spikes that are entrained by the master, while ignoring slave spikes that are not so entrained. Thus it is more appropriate for a forced system than for a mutually coupled system.

In the run of Figure 4, the spike correlation we found was 57 per cent, while the same run with no forcing showed spike correlation only 7 per cent.

Chaos Synchronization in Networks

Goodwin also considered the global economy as a massively complex dynamical network of chaotic nodes. In this context, the emergence of a large clique of nodes in

spontaneous synchrony, like a school of fish or flock of birds, may function as a conspiracy, without any explicit manipulation. This possibility, a sort-of pseudoconspiracy, actually occurs in computer simulations of complex dynamical networks of identical nodes, such as a reaction-diffusion lattice of Rössler schemes, as given in the subsequent section.

Simulation Results

Thanks to the spectacular decline in the cost of massively parallel super-computers, we are able to present here the result of long simulations of two-dimensional lattices of Rössler schemes. These have square arrays of 400 by 400 nodes, each coupled by diffusion of the z -variable from its four nearest neighbours. This means, approximately, that the two-dimensional laplacean of z at a given node modulates the control parameter b in equation (3) above. The response diagram of the Rössler scheme, with b in the range $[0, 2]$, is shown in Figure 2. Note that the chaotic attractors with b around 0.2 are bracketed by three-periodic windows.

The nodes begin with random initial conditions, chosen in the domain, $u \in [-10, 10]$, $v \in [-10, 10]$, $z \in [0, 20]$ and all have the same value of the control parameters, $a = b = 0.2$ and $c = 5.7$, in the chaotic regime. Thus, there is no master bank manipulating the controls and sync develops solely from mathematical cooperation of the policy decisions of the nodes. The diffusion rate in the z direction is chosen in the range, $D \in [0, 5]$.

This corresponds to coupling the schemes:

$$u' = au + v \tag{4}$$

$$v' = -u - z \tag{5}$$

$$z' = b + (v - c)z + d \tag{6}$$

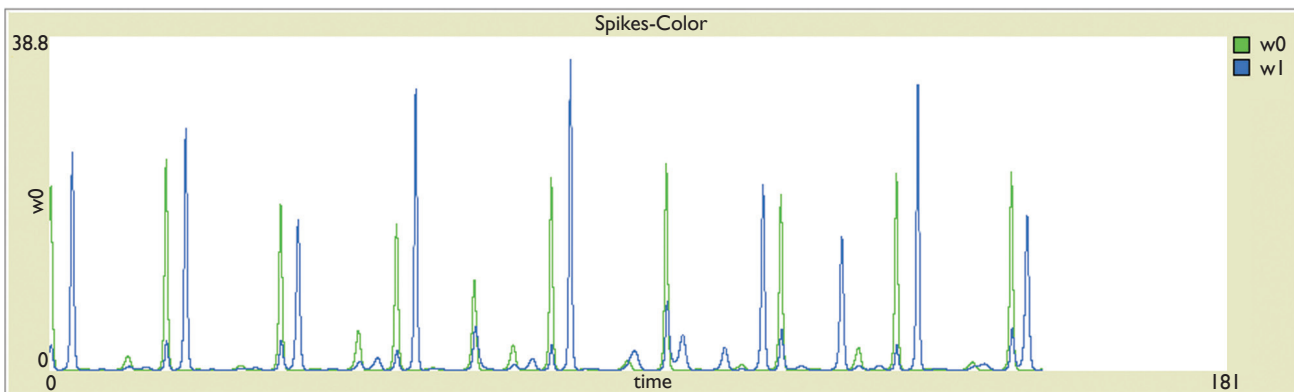


Figure 4. Rössler forced Rössler, Master (green) and Slave (blue), forcing constant 2.0, showing strong synchronization. The simulate time shows is from 1000 to 1150.

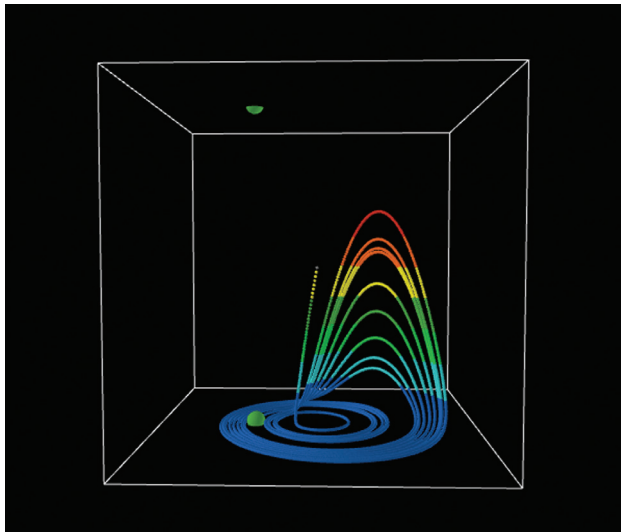


Figure 5. Colour code projected onto a single Rössler attractor

where d is now the control parameter, linked to the z values of the four nearest neighbours by diffusion. In other words, the b parameter is modulated around a base value of 0.2 by the diffusion factor, d . The boundary conditions in this simulation are periodic, that is, the lattice resides on a torus.

The visualization strategy is to show the instantaneous states of the z -value at each node as a colour: from blue (at $z = 0$) to red (at $z = 25$). Where we see a blue zone, a ring or disc for example, we are observing phase synchrony in the z variable, the one with the hook, or spike, behaviour. The colour code is shown in Figure 5.

We may understand this phase coherence and the variation of synchronous phases across the two-dimensional lattice of nodes, by means of Zeeman’s visualization strategy.¹⁴ We imagine the lattice as a square paper napkin. At a given instant in the simulation, the scheme at each node has instantaneous values of (u, v, z) and also d . Thus we may map the paper napkin into the standard cell, the four-space of (u, v, z, d) .

At the start, the article is very crumpled up and positioned randomly in the cell. As the simulation advances step-by-step, the napkin is moved about in the cell and all image points are attracted to the locus of the Rössler attractor in the cell. The colour code lives in the cell, as layers of different colour, parameterized by z . As the folded napkin image moves about in the cell, the colours it moves through

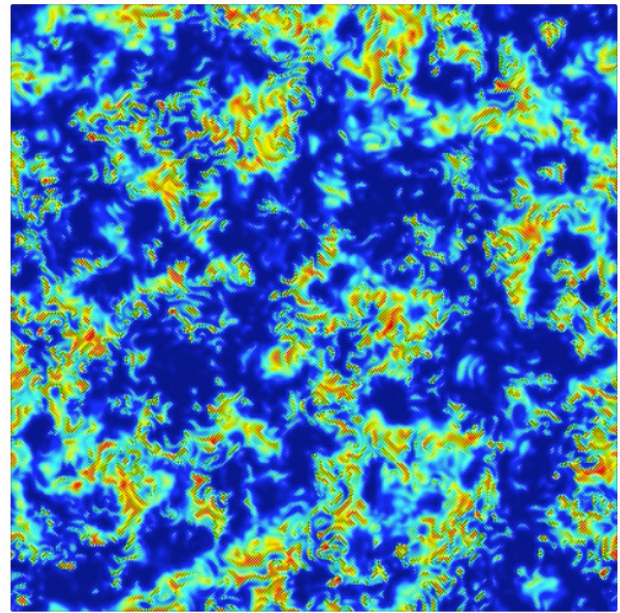


Figure 6. Single frame, 400 x 400 lattice of Rössler attractors, Z-forcing = 4.0

are pulled-back to the original lattice and these we see in the simulation as a movie.

A single frame of this movie is shown in Figure 6. But different blue zones, for example, may be in phase sync with phase difference 360 degrees. Thus image points in between are pulled around the attractor ring and are stuck in phase sync of relative phases between 0 and 360 degrees. Our simulations are recorded as movies, each frame representing an instantaneous state of the entire lattice, one node per pixel. Thus a blue node, in a single frame, is frozen in an instant of an actual (u, v) nearly periodic motion in the (u, v) plane, mostly with low (blue) z . The z value has a spike-like behaviour, like a neuron but not as sharp. So the blue zones are analogous to the Chladni nodal curves (quiet zones, low z) of a vibrating membrane, while the red zones are roughly synchronous spikes, analogous to the antinodes of a vibrating membrane.

In the movie there is an apparent near-periodicity of the full images. To reveal this we computed the image entropy of each frame. This is a measure of image complexity, similar to the fractal dimension.¹⁵ A plot of the image entropy versus time is shown in Figure 7, where time varies from 500 to 1000.

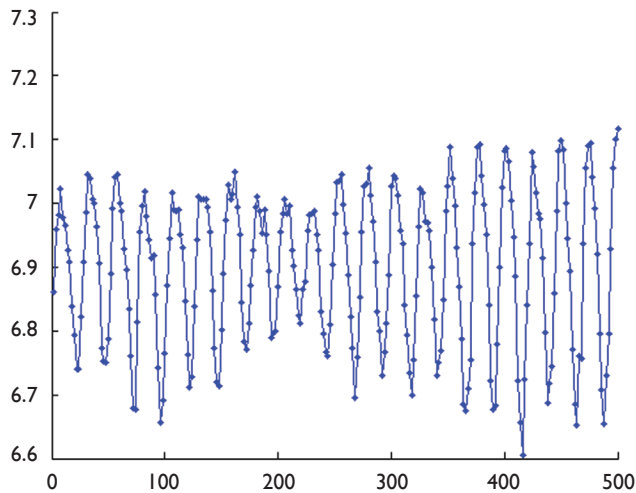


Figure 7. Image entropy of frames of the simulation movie, run h6

Speculations on the Global Economy

Our experiments on a 2D lattice of Rössler attractors are inspired partly by pure mathematics and partly by recent research on the nonlinear dynamics of the human heart, using 2D coupled lattices of oscillators.¹⁶ Goodwin's suggestion of 1990 to regard the global economy as a complex dynamical system probably requires a neural network model with long links, rather than a biological organ model like the heart, which has only nearest-neighbour connections. Nevertheless, we may regard our Rössler lattice as a simple first step in the direction of understanding the behaviour of the global economy as a complex dynamical network of chaotic elements. Along this line, then, we may give an economic interpretation to our simulation results.

The chief feature of dynamic behaviour of our lattice is that of the propagating red islands of anti-nodal (tight peak or spike) synchronization, in a blue sea of nodal (loose oscillatory or predator-prey) synchronization. This feature, which may look like a golf-course conspiracy, is actually of purely mathematic origin. It is an emergent feature of a complex dynamical system.

In the human heart, red islands in a blue sea might correspond to arrhythmias such as atrial or ventricular fibrillation. Recent studies suggest that red islands must satisfy certain size restrictions in order to propagate and may then trigger the fatal fibrillatory state.¹⁷

It is not impossible that models such as this one might inform efforts to stabilize, or even to destabilize, the global economic network. The present troubles of the global economy, especially in the United States and Europe,

may actually be evidence of a propagating red node, that is, an economic fibrillation.

Conclusion

Richard Goodwin foresaw the advent of network economics in 1990:

The modern economy consists of a great variety of separate activities intimately linked, directly or indirectly through markets, with all or most of the other sector.¹⁸

Our simulations of massively complex dynamical systems composed of the models suggested by Richard Goodwin for chaotic economic systems suggest that pseudo-conspiracies may develop from mathematical reasons alone. Further, instabilities of our global economic system may be better understood from the further study of these Goodwin networks through extensive simulations. We plan to pursue this direction in future work.

Acknowledgements

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Notes

1. Submitted to the *IIM Kozhikode Society & Management Review*.
2. The highlights are described in Abraham and Shaw (1992, ch. 4).
3. See (Abraham & Shaw, 1992, ch. 5) and (Strogatz, 2003, ch. 4).
4. See (Abraham, 1984).
5. Personal communication via Doyne Farmer (1986), reported in Stone (1992).
6. See Xie *et al.* (2007), Sato *et al.* (2009) and references therein.
7. An excellent text is Mitchell (2009).
8. See, for example, Abraham and Shaw (1992) and Abraham (1997).
9. Here we have adopted the notation of Goodwin (1990, pp. 48, 66), but write z for his k .
10. From Wikipedia entry for *Rössler attractor*.
11. Goodwin (1990) p. 87.
12. Goodwin (1990, final chapter).
13. Zeeman (1977, p. 117).
14. See Abraham (2012) for the definition.

15. For example, see Sato *et al.* (2009).
16. See Xie *et al.* (2007) and Sato *et al.* (2009, p. 2985).
17. Final page, Goodwin (1990, p. 130).

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