# A Note on a Bernoulli Demand Inventory Model

**Pritibhushan Sinha**

In this paper we present a single-item, continuous monitoring inventory model with probabilistic demand for the item and probabilistic lead time of order replenishment. It is assumed that demand follows Bernoulli distribution and an order is placed after stock becomes zero. The model is amenable to exact analysis and the optimal order quantity has a closed-form solution. Also, it is shown that this solution is the basic deterministic economic order quantity formula applied with average demand rate, if it is further assumed that the lead time is zero. Cost of an optimal solution of the model may be used as an upper bound for the same of some other more general policies.

**Key Words:**

Bernoulli Distribution, Inventory Model

## Introduction

Inventory decisions such as when to order and how much to order for different items of consumption or sales are indeed very important. These affect financial performance of a firm through capital being tied up due to lost sales or production. Delivery delays or stock outs caused by such decisions may affect customer satisfaction. Inventory analysis has a long history and one of the first models to be discussed was the basic economic order quantity (EOQ) model (see, Silver et al. 1998). The model assumes constant demand rate and constant lead time. It is cited frequently in the relevant literature and many variants of the model have been considered, as in Teng et al. ( 2005) to broaden the scope of application of the model. Also, the use of EOQ formula is quite common in practice with the understanding that it would give a good approximation, if not an optimal solution. Probabilistic inventory models with probabilistic demand and supply are more appropriate in many real situations. But, such models also pose greater difficulty in analysis and often become intractable. Sometimes EOQ is used as an approximation in such instances. The correctness of EOQ foiinula in probabilistic models has been a point of investigation as exemplified by Axsater (1996). In some cases, EOQ is a bound for the optimal order quantity,

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whereas in some others EOQ is the limiting solution as variance in demand tends to zero. In the present article, we propose an inventory model with probabilistic demand and lead time and show its relation with EOQ. In this model, demand follows a Bernoulli distribution and lead time follows a general discrete distribution.

Dunsmuir and Snyder (1989) and Janssen et al. (1998) discuss probabilistic inventory models with demand following a compound Bernoulli process. During a time unit the demand is positive with a fixed probability (0 < p 1), and given the demand is positive it follows a general distribution F(.). They consider a *(R, s, Q)* model (R - review period, *s* - reordering stock position, *Q* - order quantity) and derive approximate solutions for system parameters as service level and average physical stock. Approximate solutions obtained in Dunsmuir and Snyder (1989) are improved in Janssen et al. (1998). Such improvements are also established empirically. Janssen et al. (1998) use a constraint on the service level and subject to such a constraint, optimal values for the policy parameters *s* and *Q* are obtained for a given *R*. But, as proposed by Moon and Choi (1994), Janssen et al. (1998) use a distribution free approach in the optimization problem. In this approach, an upper bound of the objective function and not the exact objective function is minimized. Although this is a useful and practical approach in the absence of complete distributional information, it involves an approximation. Janssen et al. (1998) also give some examples of practical situations where the assumption of compound Bernoulli process demand would hold.

' In the present exposition, we consider an inventory model with Bernoulli demand and a continuous monitoring policy. Orders are placed when inventory position becomes zero, i.e., $s = 0$. In Bernoulli demand, the demand in a time unit is 1 with probability *p* and is 0 with the probability (1 –p). Bernoulli demand may be considered as a special case of compound Bernoulli demand. But in the present model, an order is placed only after the previous order has been received and all units of the replenishment have been consumed. There can be a maximum of one pending order at any point in time. This is not the situation in an *(R,* s, Q) model for which there may be more than one pending order. Further, we include a shortage cost which is proportional to the number of units of shortage. With such assumptions, the model becomes amenable to exact analysis. A closed-form solution for the order quantity Q can be obtained. It is also seen that this solution is a probabilistic version of the EOQ in which the average demand rate is substituting for the constant demand rate in the special case of no lead time. The model proposed by us has not been considered in extant literature.

Though the assumption of Bernoulli demand pattern is somewhat restrictive, the model would be a good approximation in a number of situations. For example, the assumption may be reasonably accurate for a retail business. In this case, there may be items such that, in most of the occasions of sales only one unit is purchased. Lead time of supply may be small allowing the dispatch of an order after stock becomes zero. Another application could be found in slow-moving maintenance spares items. The demand rate may be small and very rarely there may be a demand of more than one unit at a time. The probability of a demand arising during lead time should-also be very small. Examples of such maintenance spares are bearings, belts etc. for machines such as pumps and motors. More generally, the model should be applicable when expected cost of shortages during lead time is relatively much less with respect to ordering and inventory holding costs. This may happen for one or some combination of the reasons such as, small lead time, very low average demand per unit time, and small cost for per unit of shortage.

The paper is organized as follows: In Section 2, `Model and Analysis,' the model is described and analytical formulae are derived. A numerical example relevant to the model is also given in this section. In Section 3, scope of the model and further extension possibilities are discussed.

**Model and Analysis**

Notations:

The following notation is used in this section and later.

p : Probability of demand being one unit during a unit of time;

$Q$ : Order quantity, $Q = 0,1, ...$;

L : Average lead time;

c : Cost of shortage per unit;

h : Inventory holding cost per unit, per unit time;

A : Ordering cost for an order;

r : Profit per unit sold (over manufacturing and other such proportional costs);

$X\sim$ : Random variable denoting demand during the i-th time unit, in a cycle. $\boldsymbol{X} = 1$ or 0;

$Y$ : Random variable representing lead time of an order. $Y = 0, 1, ...$ ;

C : Discrete random variable denoting total cost in a cycle;

$T$ : Random variable denoting time length of a cycle. $T = Q, Q + 1, \ldots$;

T' : Time, random variable, after which inventory position becomes 0 from Q, in a cycle. T = Q, Q + 1, ...;

Assumptions in the Model:

We make the following assumptions in the model.

i) The system starts at time = 0 with an inventory of Q units.

ii) Demand for the item in every unit of time interval follows a Bernoulli distribution with parameter $p$. That is, $X_i = 1$ with probability $p$ $(0 < p$ 1) and $X = 0$ with probability (1 – p). The demand occurs at the end of the interval.

iii) Demands at different unit time intervals are mutually independent.

iv) As the inventory position becomes zero, an order of Q units is placed and is received after $Y$ time units, the lead time. $Y$ has average $L$ *(e" 0)*. Lead times are mutually independent and independent of demands.

v) Demand during a lead time, including the demand, if any, in the last unit time interval in a lead time is unsatisfied and cost of shortage is c / unit of shortage (c e" 0). Such shortage cost is a measure of goodwill loss, customer dissatisfaction etc.

vi) Inventory holding cost $h$, ordering cost A, profit per unit sold r are non-negative ( $h$, A, r 0).

vii) Objective function considered is long term cost per unit

time. Analysis:

First, we note that the inventory position follows a renewal process. Each receipt of an order is a renewal point. The same process restarts probabilistically at every such point. Each realization of the process, from one such point till the next is an identical and independent renewal cycle. The discrete random variable C denotes the total cost, considering profits from sold units, costs of inventory holding, ordering and shortage in a cycle. The discrete random variable T denotes the time of a cycle,

which is the time between two receipts of orders. The random variable **r,** as defined earlier, gives the number of unit time intervals at the end of which stock becomes zero after an order is received in a cycle. T Q and it is not difficult to see that T follows a Pascal distribution, also called negative binomial distribution, with the following properties,

$$\text{Probability } \{T' = x1 =^{x-1} C_{Q-}, p^{Q} (1 - p)^{' -Q}, x = Q, Q + 1, \ldots \qquad (2.1)$$

(2.2)

of T, E[T] = Q / p .2)

$T = T' + Y$. E[T] = $Q/p + L$ is the average time length of a cycle. Let, $g(Q)$ denote the average inventory holding cost per cycle, when the order quantity is Q. Since demands for unit time intervals are identical and average time for one unit of demand is 1/p, we can write,

$$g(Q) = \frac{Qh}{p} + g(Q - 1), \quad Q \geq 2, \text{ integer,} \tag{2.3}$$

and g(1) = $h/p$. We, thus, get,

$$g(Q) = \frac{h}{2p} Q(Q + 1), \quad Q \geq 1, \text{ integer,} \tag{2.4}$$

Expected cost for shortages in a cycle is $cLp$. So, expected cost in a cycle,

$$E[C] = -Qr + A + \frac{h}{2p} Q(Q + 1) + cLp, \quad Q \geq 1, \text{ integer,} \tag{2.5}$$

Applying renewal reward theorem (see, Ross 1990) cost per unit time in the long run is E[C] / E[T]. Also, with the observation that for $Q = 0$ (which means there is no ordering, the item not being inventoried at all) there is no inventory or ordering cost and every cycle has expected cost of shortages only, long run cost per unit time is given as,

$K(Q) = cp, \; Q = 0;$

$$= \left(-Qr + A + \frac{h}{2p} Q(Q + 1) + cLp\right) / \left(\frac{Q}{p} + L\right), \quad Q \geq 1, \text{ integer,} \tag{2.6}$$

To minimize K(Q), we take it as an unconstrained problem and consider $M(Q) = \left(-Qr + A + \frac{h}{2p} Q(Q + 1) + cLp\right) / \left(\frac{Q}{p} + L\right), \; -\infty < Q < \infty$

and the derivatives dM/dQ & $d^2M/dQ^2$.

$$dM/dQ = \left(\frac{Q^2 h}{2p} + QLh + L\left(\frac{h}{2p} - (r+c)\right) - \frac{A}{p}\right) / \left(\frac{Q}{p} + L\right)^2. \tag{2.7}$$

$$d^2M/dQ^2 = \left(\frac{L^2h}{p} \frac{2L}{p}\left(\frac{h}{2p} - (r + c)\right) + 2A\right) / (Q + L)^3 \qquad (2.8)$$

If $\frac{Lh}{2} + (r + c) + \frac{A}{Lp} > \frac{h}{2p}$ then $d^2M/dQ^2 > 0$ for $Q > 0$ so that $M(Q)$ is strictly convex for $Q > 0$. Thus, there is a unique minimum $Q^* > 0$, if there is any positive, finite minimum, of $M(Q)$ in such a case. The minimum $Q^*$ can be found out by solving, $\frac{Q^2 h}{2p} + \frac{QLh}{p} + L\left(\frac{h}{2p} - (r + c)\right) - \frac{A}{p} = 0$. This gives,

$$- Lp + \sqrt{(Lp)^2 + \frac{2p^2}{Ah}\left(L(r + c) - \frac{h}{2p}\right) + \frac{2}{p}} \qquad (2.9)$$

If $Q^* > 0$, the optimal order quantity $(Q_{opt})$ is then found by verifying the costs at the next higher and lower integers to $Q^*$, and comparing with the cost per unit time for $Q = 0$, K(0). Minimum cost solution is taken as $Q_{opt}$. If $Q^* < 0$, $Q_0$ is fixed at zero. $Q_{opt} = 0$ would imply that the item is not profitable to be stored, and no order is placed. It may also be noted that, when $L = 0$, $Q^* = \sqrt{2pA/h}$, the basic EOQ with the average demand rate (p) substituting the deterministic demand rate.

If $\frac{Lh}{2} + (r + c) + \frac{A}{Lp} \frac{h}{2p}$ — $M(Q)$ is concave. Also, $M(0) = A/L + cp$ and $M(Q) \to \infty$ as $Q \to \infty$ (assuming $h > 0$; which must hold if at least one of r, c and A is positive, in this case). So, for $M(Q)$ concave, $Q = 0$ minimizes $M(Q)$ and also $K(0) \leq M(0)$. Thus, $M(Q)$ concave would again imply that the item should not be stored $(Q_{opt} = 0)$.

A sufficient condition for $Q_{opt} > 0$ is, $K(0) > K(1)$, i.e., $cp > h$ 1 $(\frac{r}{2} + \frac{A}{} + P + clip) / (p + L)$, which gives $(r + c) > (A + hip)$. This has the simple interpretation that, effective contribution from one unit sales is greater than the ordering and average inventory cost in a cycle, when ordering quantity is one. This condition also would imply $M(Q)$ convex, but $M(Q)$ may be convex otherwise too. $M(Q)$ is strictly convex, for $Q > 0$, if and only if $(^r + {}^c) > \frac{h\ Lh\ A}{2p\ 2\ Lp}$ - This gives a lower limit for effective contribution of one unit of sales below which not storing the item would be the best solution.

A Numerical Example:

Here, we discuss a numerical illustration of the application of the model. Suppose the demand for an item of a particular brand in a retail store is 1 with probability 0.1 and 0 with probability 0.9, in an hour; and other conditions of the model hold. The average demand rate is 1 / day, assuming it operates 10 hours a day. Other data are taken as, r = Rs 10 / unit, A = Rs 100, h = Rs 0.006 / unit / hour. Calculations are done for two values of shortage cost c, Rs 5 /unit and Rs 10 /unit. Optimal order quantity $(Q_{0,,})$, long term cost per unit time for the optimal order quantity $(K(Q_{op1}))$ for different values of average lead time $L$ are shown in Table 1. A plot of $K(Q)$ with $Q$ with varying L & c is shown in Figure 1. Though the value for c may be subjective to some extent, the values for the parameters taken here are realistic. If one assumes that the cost for a unit is Rs 100, and inventory holding cost is 18% per year, then h = Rs 0.006 / unit / hour (1 year = 300 x 10 = 3000 hours).

**Table** 1: **Solutions for the Numerical Example**

| **Ohs. No.** | **Average Lead** Time (hr.) | **Q*** | **Q o,,** | $K(Q_{opt})$ (Rs/hr.) |
|---|---|---|---|---|
| c = 1 0 | | | | |
| 1. | 70 | 75.87 | 76 | -0.5415 |
| 2. | 30 | 66.54 | 67 | -0.5976 |
| 3. | 20 | 63.83 | 64 | -0.6139 |
| 4. | 10 | 60.91 | 61 | -0.6315 |
| 5. | 5 | 59.35 | 59 | -0.6408 |
| 6. | 0 (EOQ) | 57.73 | 58 | -0.6506 |
| c = 5 | | | | |
| 1. | 70 | 82.64 | 83 | -0.5009 |
| 2. | 30 | 70.05 | 70 | -0.5766 |
| 3. | 20 | 66.31 | 66 | -0.5990 |
| 4. | 10 | 62.24 | 62 | -0.6235 |
| 5. | 5 | 60.04 | 60 | -0.6367 |
| 6. | 0 | 57.73 | 58 | -0.6506 |

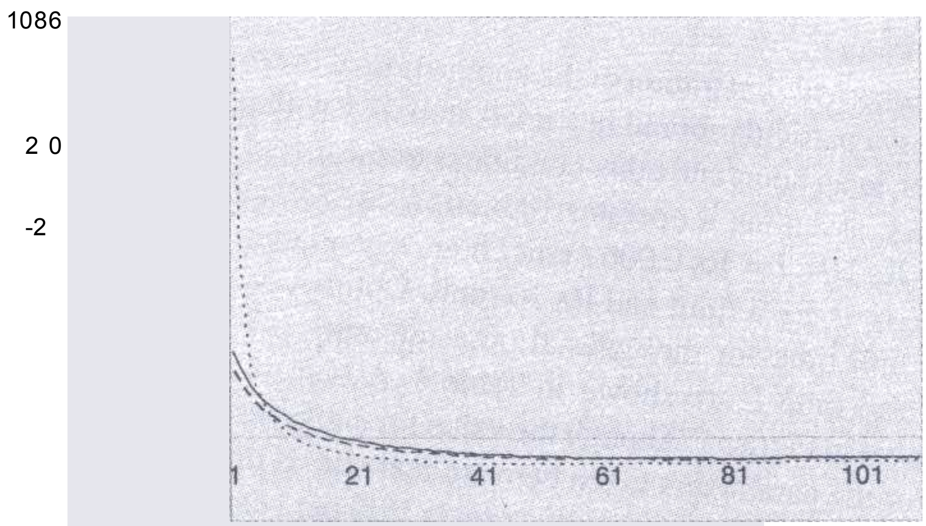


**Figure 1: Long Term Cost per Unit Time for Different Values of the Parameters**

As can be seen in this case, although cost for an optimal solution decreases with decreasing average lead time, the effect is not very significant. For c = Rs 5 /unit, by decreasing the average lead time from 70 hours to 30 hours, cost per unit time decreases by 9.8%, and $Q_{om}$ decreases by 11.8%. For increased c = 10 /unit, for the same decrease in average lead time, cost per unit time decreases by 15% and $Q_0$ has a decrease of 15.7%. Optimal order quantity changes relatively slowly with average lead time. Similar pattern is also observed as values of other parameters are changed (Table 2 — 5). In these tables, per cent increase in cost (of an optimal solution) compared with the same with lead time zero is shown for the largest average lead time. This increase is not very high, particularly for low shortage cost per unit (c) and low inventory holding cost (h). In such situations, solution given by the model should compare favourably with that of other similar models.

**Table 2: Solutions with Varying Probability of Demand (p)**

| Obs. No. | Average Lead Time (hr.) | $Q_{opt}$ | $K(Q_{opt})$ (Rs/ hr.) |
|---|---|---|---|
| p=0.05 (c = 7.5, h = 0.006, A 100, r= 10) | | | |
| 1. | = 70 | 48 | -0.2064 (18.1%) |
| 2. | 30 | 44 | -0.2307 |
| 3. | 20 | 43 | -0.2375 |
| 4. | 10 | 42 | -0.2446 |
| 5. | 5 | 41 | -0.2483 |
| 6. | 0 | 41 | -0.2521 |
| *p=0.2 (c=7.5,* h=0.006,A= 100, *r* = 10) | | | |
| 1. | 70 | 138 | -1.1675 (22.5%) |
| 2. | 30 | 111 | -1.3308 |
| 3. | 20 | 102 | -1.3819 |
| 4. | 10 | 93 | -1.4397 |
| 5. | 5 | 88 | -1.4720 |
| 6. | 0 | 82 | -1.5071 |

**Table 3: Solutions with Varying Inventory Holding Cost (h)**

| Obs. No. | Average Lead Time (hr.) | | $K(Q_{opt})$ (Rs/ hr.) |
|---|---|---|---|
| h=0.003 (p=0.1, c=7.5, A= 100, r = 10) | | | |
| 1. | 70 | 115 | -0.6536 (13.3%) |
| 2. | 30 | 98 | -0.7049 |
| 3. | 20 | 93 | -0.7199 |
| 4. | 10 | 87 | -0.7360 |
| 5. | 5 | 85 | -0.7446 |
| 6. | 0 | 82 | -0.7536 |
| h=0.012 (p=0.1, c=7.5, A= 100, r = 10) | | | |
| 1. | 70 | 54 | -0.3413 (31.9%) |
| 2. | 30 | 47 | -0.4243 |
| 3. | 20 | 45 | -0.4485 |
| 4. | 10 | 43 | -0.4750 |
| 5. | 5 | 42 | -0.4892 |
| 6. | 0 | 41 | -0.5041 |

**Table 4: Solutions with Varying Ordering Cost (A)**

| Obs. No. | Average Lead Time (hr.) | $Q_{opt}$ | $K(Q_{opt})$ (Rs/ hr.) |
|---|---|---|---|
| A=50(p=0.1,c=7.5,h=0.006,r=10) | | | |
| 1. | 70 | 69 | -0.5824 (22.5%) |
| 2. | 30 | 55 | -0.6640 |
| 3. | 20 | 51 | -0.6895 |
| 4. | 10 | 46 | -0.7184 |
| 5. | 5 | 44 | -0.7345 |
| 6. | 0 | 41 | -0.75201 |
| A=200(p=0.1,c=7.5,h=0.006,r=10) | | | |
| 1. | 70 | 97 | -0.4135 (18.0%) |
| 2. | 30 | 89 | -0.4643 |
| 3. | 20 | 86 | -0.4779 |
| 4. | 10 | 84 | -0.4921 |
| 5. | 5 . | 83 | -0.4995 |
| 6. | 0 | 82 | -0.5071 |

**Table 5: Solutions with Varying Profit per Unit (r)**

| Obs. No. | Average Lead Time (hr.) | $Q_{opt}$ | $K(Q_{opt})$ (Rs/ hr.) |
|---|---|---|---|
| r=5 (p=0.1, c=7.5, h=0.006,A= 100) | | | |
| 1. | 70 | 72 | -0.0631 (58.1%) |
| 2. | 30 | 65 | -0.1085 |
| 3. | 20 | 63 | -0.1215 |
| 4. | 10 | 60 | -0.1356 |
| 5. | 5 | 59 | -0.1429 |
| 6. | 0 | 58 | -0.1506 |
| r=20 (p=0.1, c=7.5,h=0.006,A= 100) | | | |
| 1. | 70 | 92 | -1.4453 (12.4%) |
| 2. | 30 | 75 | -1.5468 |
| 3. | 20 | 70 | -1.5776 |
| 4. | 10 | 64 | -1.6119 |
| 5. | 5 | 61 | -1.6306 |
| 6. | 0 | 58 | -1.6506 |

The average demand rate or the probability p may be estimated with the sample average of demands of some unit time intervals. One way to test the independence of demands may be to test the equality proportions of two Bernoulli populations. One proportion would be of the unit time intervals having a non-zero demand, succeeding unit intervals with non-zero demand. The other would be of such unit intervals, but following unit intervals having zero demand. In case of independent demands these two proportions should be the same.

## Discussion and Conclusion

We have presented a probabilistic inventory model with Bernoulli demand and general discrete random lead time. An order is placed only after stock becomes zero. The model is limited because it considers such demand distribution and for the condition as mentioned. But there can be some practical situations where the model may be applicable particularly because now small lead times are often possible by virtue of improved logistics and communication systems. An exact optimal solution expressed with a closed-form formula, can be found with the model.

A more general model is an (s, Q) type model, allowing s 0, other conditions remaining unchanged. When there is no lead time, the present model is equivalent to such an *(s, Q)* model. If the expected shortage cost per cycle is relatively less compared with other costs, optimal solution given by the model should compare favourably with that of the (s, Q) model. For all cases, cost of an optimal solution of the present model would be an upper bound for that of a (s, Q) model, or for other further generalized policies (when long term cost per unit time is defined for such cases). As such, it may be used for verifying results obtained by simulation experiments / approximate methods for such models, for which exact analyses are not available. It will surely be very useful if the present model may be generalized with exact analysis to such models.

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1